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Existence and Accuracy Results
for
Composite Dilation Wavelets

by

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Dedication

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Chapter 1

Introduction

This dissertation answers some existence questions for composite dilation wavelets generating orthonormal bases for $L^2(\mathbb{R}^n)$. In this first chapter we cover some necessary background information from affine systems to finite reflection groups to accuracy of multidimensional functions. In chapter two, we begin by proving some sufficient conditions for the existence of minimally supported frequency composite dilation wavelets. We then prove the existence of minimally supported frequency composite dilation wavelets in every dimension. We show that they exist for arbitrary lattices and every group with a very minor restriction to their fundamental domain. In chapter two, we also prove the existence of such wavelets with a single generator. We conclude chapter two by answering similar questions for composite dilation wavelets generating Parseval frames for $L^2(\mathbb{R}^n)$. Chapter three examines composite dilation wavelets in the time domain. The main portion of chapter three involves finding a compactly supported composite dilation wavelet with greater accuracy than the Haar wavelet. Along the way, we look at the accuracy of composite dilation systems and establish the necessary conditions for higher accuracy composite dilation wavelets.

1.1 Composite Dilation Wavelets

In both pure and applied mathematics, there is significant interest in the study of representations of multidimensional functions. The considerable literature on wavelets has come from this pursuit and led to noteworthy advances. Wavelets, however, are rigid in that they must maintain their orientation throughout the basis. Composite dilation wavelets are, in a sense, a generalization of wavelets that provides the freedom to manipulate the orientation of the functions. This has significant value in both applications and pure mathematics.

Here we examine affine systems in a general context, see how wavelets fall in this category, and discuss the newer subset of affine systems called composite dilation systems. We examine the multiresolution analysis associated with each type of system and discuss some parallels. We briefly examine some properties of the dilation and translation operators that generate the affine systems while simultaneously establishing notation to be used throughout the discussion.

1.1.1 Affine Systems, Wavelets, and Composite Dilation Wavelets

Affine systems are formed by applying two kinds of unitary operators to a set of functions. These operators are dilations and translations.

Definition 1.1. For $a \in GL_n(\mathbb{R})$ and $f \in L^2(\mathbb{R}^n)$, the *dilation of f by a* is the operator $D_a : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ such that

$$D_a f(x) = |\det(a)|^{-\frac{1}{2}} f(a^{-1}x).$$

Definition 1.2. For $k \in \Gamma$, Γ a full rank lattice ($\Gamma = c\mathbb{Z}^n$ for $c \in GL_n(\mathbb{R})$), and

$f \in L^2(\mathbb{R}^n)$, the *translation of f by k* is the operator $T_k : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ such that

$$T_k f(x) = f(x - k).$$

Then, for a countable subset of invertible matrices $C \subset GL_n(\mathbb{R})$, Γ a full rank lattice, and $\psi^1, \dots, \psi^L \in L^2(\mathbb{R}^n)$, the *affine system produced by C , Γ , and $\Psi = (\psi^1, \dots, \psi^L)$* is the set

$$\mathcal{A}_{C\Gamma}(\Psi) = \{D_c T_k \psi^l : c \in C, k \in \Gamma, 1 \leq l \leq L\}. \quad (1.1)$$

When $C = \{2^j : j \in \mathbb{Z}\}$, $\Gamma = \mathbb{Z}$, and $\psi \in L^2(\mathbb{R})$, then (1.1) can be written as the familiar system

$$\{\psi_{j,k}(x)\}_{j,k \in \mathbb{Z}} = \left\{2^{-\frac{j}{2}} \psi(2^{-j}x - k) : j, k \in \mathbb{Z}\right\}. \quad (1.2)$$

In this case, we recall that ψ is a *wavelet* when the collection of functions (1.2) is an orthonormal basis for $L^2(\mathbb{R}^n)$. In a more general situation, ψ is a *Parseval frame wavelet* when (1.2) is a Parseval frame for $L^2(\mathbb{R}^n)$.

In [9] and [10], Guo, Labate, Lim, Weiss, and Wilson introduce affine systems with composite dilations. These are affine systems obtained from (1.1) when the countable set C is the product of two, not necessarily commuting, sets of invertible matrices: $C = AB$, $A, B \subset GL_n(\mathbb{R})$. Obviously, $D_a D_b f(x) = |\det(a)|^{-\frac{1}{2}} D_b f(a^{-1}x) = |\det(b)|^{-\frac{1}{2}} |\det(a)|^{-\frac{1}{2}} f(b^{-1}a^{-1}x) = |\det(ab)|^{-\frac{1}{2}} f([ab]^{-1}x) = D_{ab} f(x)$. Therefore, for affine systems with composite dilations we have

$$\mathcal{A}_{AB\Gamma}(\Psi) = \{D_a D_b T_k \psi^l : a \in A, b \in B, k \in \Gamma, 1 \leq l \leq L\}. \quad (1.3)$$

These systems are more general than the traditional wavelet systems. When

$A = \{2^j : j \in \mathbb{Z}\}$ and B is the trivial group, then $C = A = \{2^j : j \in \mathbb{Z}\}$ and with $\Gamma = \mathbb{Z}^n$ we obtain (1.2). Therefore, traditional wavelets are a special class of affine systems with composite dilations. In this spirit, we say that Ψ is a composite dilation wavelet if the system $\mathcal{A}_{AB\Gamma}(\Psi)$ is an orthonormal basis for $L^2(\mathbb{R}^n)$.

In this paper, we study the situation where A is a set obtained by taking integer powers of some invertible, expanding matrix $a \in GL_n(\mathbb{R})$; $A = \{a^j : j \in \mathbb{Z}\}$. Furthermore, we study the situation where B is a finite group of invertible matrices; $B \subset GL_n(\mathbb{R})$ and $|B| < \infty$ where $|B|$ is the order of the group B . When B is finite, it is necessarily the case that for all $b \in B$, $|\det(b)| = 1$. Therefore, for $f \in L^2(\mathbb{R}^n)$ and B a finite subgroup of $GL_n(\mathbb{R})$, $D_b f(x) = f(b^{-1}x)$. Therefore, we make the following definition:

Definition 1.3. $\Psi = (\psi^1, \dots, \psi^L) \subset L^2(\mathbb{R}^n)$ is an *AB Γ Composite Dilation Wavelet* if the set

$$\begin{aligned} & \{D_a^j D_b T_k \psi^l : j \in \mathbb{Z}, b \in B, k \in \Gamma, l = 1, \dots, L\} \\ &= \left\{ |\det(a)|^{-\frac{j}{2}} \psi^l(b^{-1}a^{-j}x - k) : j \in \mathbb{Z}, b \in B, k \in \Gamma, l = 1, \dots, L \right\} \end{aligned}$$

is an orthonormal basis for $L^2(\mathbb{R}^n)$.

Since the groups A and B and the lattice Γ are almost certainly going to be obvious from the context, we usually suppress the $AB\Gamma$ and simply refer to Ψ as a composite dilation wavelet. In section 2.5, we will also discuss composite dilation Parseval frame wavelets. As in the traditional wavelet case, this is simply a relaxation of the orthonormal basis to a Parseval frame in definition 1.3. That is, we call Ψ a composite dilation Parseval frame wavelet if $\{D_a^j D_b T_k \psi^l : j \in \mathbb{Z}, b \in B, k \in \Gamma, l = 1, \dots, L\}$ is a Parseval frame for $L^2(\mathbb{R}^n)$.

Chapter 2 proves sufficient conditions and existence for *Minimally Supported Frequency* composite dilation wavelets. First, we recall from [11] that a *Minimally Supported Frequency* (MSF) wavelet is a wavelet, ψ , such that $|\hat{\psi}|$ is the characteristic

function of a measurable set $R \subset \widehat{\mathbb{R}}$. The motivation behind the name is that the set R supporting the MSF wavelet must satisfy $m(R) = 1$, where m is Lebesgue measure and the lattice is \mathbb{Z} . Thus, the support of $\hat{\psi}$ has the minimal measure required such that $\{T_k\psi : k \in \mathbb{Z}\}$ is an orthonormal system.

Definition 1.4. $\Psi = (\psi^1, \dots, \psi^L) \subset L^2(\mathbb{R}^n)$ is a *Minimally Supported Frequency Composite Dilation Wavelet* if $\hat{\psi}^l = |\det(c)|^{\frac{1}{2}} \chi_{R_l}$ for all $l = 1, \dots, L$ and the set

$$\{D_a^j D_b T_k \psi^l : j \in \mathbb{Z}, b \in B, k \in \Gamma, l = 1, \dots, L\}$$

is an orthonormal basis for $L^2(\mathbb{R}^n)$.

In this case, with $\Gamma = c\mathbb{Z}^n$, each set R_l must satisfy $m(R_l) = |\det(c)|^{-1} = |\det(c^{-1})|$. We call these MSF composite dilation wavelets since the support of each wavelet $\hat{\psi}^l$ has the minimal measure such that $\{D_b T_k \psi^l : k \in \Gamma, l = 1, \dots, L\}$ is an orthonormal system.

1.1.2 Composite Dilation Multiresolution Analysis

One classical method for constructing orthonormal wavelets is the well known multiresolution analysis (MRA). In traditional wavelets, for example in [11], an MRA is defined as a sequence, $\{V_j\}_{j \in \mathbb{Z}}$, of closed subspaces of $L^2(\mathbb{R}^n)$ satisfying

$$V_j \subset V_{j+1} \quad \text{for all } j \in \mathbb{Z}; \tag{1.4}$$

$$f \in V_j \text{ if and only if } f(2\cdot) \in V_{j+1} \quad \text{for all } j \in \mathbb{Z}; \tag{1.5}$$

$$\bigcap_{j \in \mathbb{Z}} V_j = \{0\}; \tag{1.6}$$

$$\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}^n); \tag{1.7}$$

$$\exists \varphi \in V_0 \text{ such that } \{T_k \varphi : k \in \mathbb{Z}\} \text{ is an orthonormal basis for } V_0. \tag{1.8}$$

In [10] an analogue to the classical MRA is introduced for affine systems with composite dilations. In this setting they develop a method for constructing composite dilation wavelets when you are given the sets of matrices A and B and the full rank lattice Γ . The authors discuss the generality for which a composite MRA can be constructed. Here we will use the case where $A = \{a^j : j \in \mathbb{Z}\}$ for an expanding matrix $a \in GL_n(\mathbb{R})$ and $B \subset GL_n(\mathbb{R})$ is a finite group. We establish the following definition:

Definition 1.5. Let $a \in GL_n(\mathbb{R})$ be an expanding matrix, $A = \{a^j : j \in \mathbb{Z}\}$, $B \subset GL_n(\mathbb{R})$ a finite group, and $\Gamma = c\mathbb{Z}^n$, $c \in GL_n(\mathbb{R})$, be a full rank lattice. An *AB Γ -MRA* is a sequence, $\{V_j\}_{j \in \mathbb{Z}}$, of closed subspaces of $L^2(\mathbb{R}^n)$ satisfying

$$(M1) \quad V_j \subset V_{j+1} \quad \text{for all } j \in \mathbb{Z};$$

$$(M2) \quad f \in V_j \text{ if and only if } f(a \cdot) \in V_{j+1} \quad \text{for all } j \in \mathbb{Z};$$

$$(M3) \quad \bigcap_{j \in \mathbb{Z}} V_j = \{0\};$$

$$(M4) \quad \overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}^n);$$

$$(M5) \quad \exists \varphi \in V_0 \text{ such that } \{D_b T_k \varphi : b \in B, k \in \Gamma\} \text{ is an orthonormal basis for } V_0.$$

Suppose the sequence is defined by $V_j = D_a^{-j} V_0$. Then, if (M1) is true, we automatically satisfy (M2). This is the usual way to construct an AB Γ -MRA. Find a function in $L^2(\mathbb{R}^n)$ such that $\{D_b T_k \varphi : b \in B, k \in \Gamma\}$ is orthonormal and create the shift invariant space determined by the semi-direct product of B and Γ . Let the closure of this space be V_0 . Then define the sequence of closed subspaces by $V_j = D_a^{-j} V_0$. Then we are only left with determining if we have satisfied (M3) and (M4).

This follows the arguments of the standard MRA (1.4)-(1.8) almost exactly. In

both cases V_0 is generally defined to be the shift invariant space determined by applying a group action to a specific generating function, φ , usually called the scaling function. The difference is that the group in the classical MRA (1.8) is the integer lattice and the group in (M5) is $B \rtimes \Gamma$, the semi-direct product.

Again, A , B , and Γ are usually clear from the context. As with the definition of a composite dilation wavelet, we generally refer to an $AB\Gamma$ -MRA as a composite dilation MRA. As this paper discusses composite dilation systems, after this section we will generally refer to a composite dilation MRA as an MRA. A composite dilation wavelet arising from an $AB\Gamma$ -MRA will obviously be an $AB\Gamma$ composite dilation wavelet. Such a wavelet will be referred to as an $AB\Gamma$ MRA composite dilation wavelet. Most of the time, we will let the reader infer A , B , and Γ from context and simply call them MRA composite dilation wavelets.

In the traditional MRA literature, much has been studied regarding different conditions satisfied by the basis of the set determined by the scaling function in (1.8). Most notably, the orthonormal basis is often relaxed to be a Parseval frame. The same can be done for a composite dilation MRA. We may replace (M5) with

$$(M5') \quad \exists \varphi \in V_0 \text{ such that } \{D_b T_k \varphi : b \in B, k \in \Gamma\} \text{ is a Parseval frame for } V_0.$$

We will identify this case by referring to such composite dilation MRAs as Parseval frame MRAs and wavelets arising from Parseval frame MRAs as MRA, composite dilation Parseval frame wavelets.

1.1.3 Notation and Preliminaries

Throughout this document, we deal with functions and their properties in both time and frequency. We will therefore transform the functions between the time and frequency domains via the Fourier transform. We adopt the notation that the time

domain is represented by \mathbb{R}^n while the frequency domain is denoted $\hat{\mathbb{R}}^n$. Elements of the time domain, denoted by letters of the Roman alphabet, will be column vectors, $x = (x_1, \dots, x_n)^t \in \mathbb{R}^n$, and elements of the frequency domain, or Fourier domain, will be row vectors, $\xi = (\xi_1, \dots, \xi_n) \in \hat{\mathbb{R}}^n$ denoted by letters of the Greek alphabet.

Definition 1.6. The space of square integral functions on a set Ω is

$$L^2(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{C} : \left(\int_{\Omega} |f(x)|^2 dx \right)^{\frac{1}{2}} < \infty \right\}$$

Although there are generalizations to other spaces, we focus on $L^2(\mathbb{R}^n)$.

We will use the Fourier transform that is a unitary operator from $L^2(\mathbb{R}^n)$ to $L^2(\hat{\mathbb{R}}^n)$ sending $f \rightarrow \hat{f}$; $\mathfrak{F} : L^2(\mathbb{R}^n) \rightarrow L^2(\hat{\mathbb{R}}^n)$. That is,

$$\mathfrak{F}[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi x} dx. \quad (1.9)$$

This is defined for functions in $L^1(\mathbb{R}^n)$ and has a natural extension to all of $L^2(\mathbb{R}^n)$. Of course, there is an inverse $\mathfrak{F}^{-1} = \mathfrak{G} : L^2(\hat{\mathbb{R}}^n) \rightarrow L^2(\mathbb{R}^n)$ sending g to \check{g} .

$$\mathfrak{G}[g](x) = \check{g}(x) = \int_{\hat{\mathbb{R}}^n} g(\xi) e^{2\pi i \xi x} d\xi. \quad (1.10)$$

The bases generated by composite dilation wavelets are defined in terms of the operators D_a^j , D_b , and T_k applied to the wavelet generating functions ψ^l for $1 \leq l \leq L$. Because of this notation, we examine how the Fourier transforms acts on these operators.

$$\begin{aligned} \mathfrak{F}[D_a f](\xi) &= [D_a f]^\wedge(\xi) = \int_{\mathbb{R}^n} D_a f(x) e^{-2\pi i \xi x} dx \\ &= \int_{\mathbb{R}^n} |\det(a)|^{-\frac{1}{2}} f(a^{-1}x) e^{-2\pi i \xi x} dx \end{aligned} \quad (1.11)$$

$$= \int_{\mathbb{R}^n} |\det(a)|^{-\frac{1}{2}} f(y) e^{-2\pi i \xi(a)y} |\det(a)| dy \quad (1.12)$$

$$= \int_{\mathbb{R}^n} |\det(a)|^{\frac{1}{2}} f(y) e^{-2\pi i (\xi a)y} \quad (1.13)$$

$$= |\det(a)|^{\frac{1}{2}} \hat{f}(\xi a) \quad (1.14)$$

In (1.11) we see that the $n \times n$ matrix a is acting on x on the left as defined by the dilation operator in 1.1. Again in (1.12), a acts on y on the left. However, in (1.13) we can interpret this as a acting on ξ on the right. The notation that \mathbb{R}^n consists of column vectors and $\hat{\mathbb{R}}^n$ consists of row vectors allows to interpret (1.12) and (1.13) in this manner. From this point of view, (1.14) motivates a definition similar to definition 1.1 that will be defined on the Fourier domain. In this case, we want an operator acting on a function in $L^2(\hat{\mathbb{R}}^n)$ that multiplies the variable by the dilation matrix on the right. This operator will not involve the inverse of the dilation matrix.

Definition 1.7. For $a \in GL_n(\mathbb{R})$ and $g \in L^2(\hat{\mathbb{R}}^n)$, the *Fourier domain dilation of g by a* is the operator $\hat{D}_a : L^2(\hat{\mathbb{R}}^n) \longrightarrow L^2(\hat{\mathbb{R}}^n)$ such that

$$\hat{D}_a g(\xi) = |\det(a)|^{\frac{1}{2}} g(\xi a).$$

Therefore, using (1.14) and definition 1.7, we have

$$\mathfrak{F}[D_a f](\xi) = [D_a f]^\wedge(\xi) = |\det(a)|^{\frac{1}{2}} \hat{f}(\xi a) = \hat{D}_a \hat{f}(\xi). \quad (1.15)$$

If $a, b \in GL_n(\mathbb{R})$ and $|\det(b)| = 1$, then

$$\begin{aligned} \mathfrak{F}[D_a^j D_b f](\xi) &= [D_a^j D_b f]^\wedge(\xi) = [D_{a^j b} f]^\wedge(\xi) \\ &= \hat{D}_{a^j b} \hat{f}(\xi) \\ &= |\det(a)|^{\frac{j}{2}} \hat{f}(\xi a^j b) \end{aligned}$$

$$\begin{aligned}
&= |\det(a)|^{\frac{i}{2}} \hat{D}_b \hat{f}(\xi a^j) \\
&= [\hat{D}_a^j \hat{D}_b \hat{f}](\xi)
\end{aligned}$$

Another useful unitary operator in the Fourier domain is *modulation*, a multiplication by an exponential function.

Definition 1.8. For $k \in \Gamma$ and $g \in L^2(\hat{\mathbb{R}}^n)$, the *modulation of g by k* is the operator $M_k : L^2(\hat{\mathbb{R}}^n) \rightarrow L^2(\hat{\mathbb{R}}^n)$ such that

$$M_k g(\xi) = e^{-2\pi i \xi k} g(\xi).$$

It is easy to verify that the Fourier transform of a translation of f by k is modulation of \hat{f} by k .

$$\begin{aligned}
\mathfrak{F}[T_k f](\xi) &= [T_k f]^\wedge(\xi) = \int_{\mathbb{R}^n} T_k f(x) e^{-2\pi i \xi x} dx \\
&= \int_{\mathbb{R}^n} f(x - k) e^{-2\pi i \xi x} dx \\
&= \int_{\mathbb{R}^n} f(y) e^{-2\pi i \xi (y+k)} dy \\
&= e^{-2\pi i \xi k} \int_{\mathbb{R}^n} f(y) e^{-2\pi i \xi y} dy \\
&= e^{-2\pi i \xi k} \hat{f}(\xi) \\
&= M_k \hat{f}(\xi).
\end{aligned} \tag{1.16}$$

Therefore, using (1.15) and (1.16) we have

$$\begin{aligned}
\mathfrak{F}[D_a T_k f](\xi) &= [D_a T_k f]^\wedge(\xi) = \hat{D}_a [T_k f]^\wedge(\xi) \\
&= |\det(a)|^{\frac{1}{2}} [T_k f]^\wedge(\xi a) \\
&= |\det(a)|^{\frac{1}{2}} M_k \hat{f}(\xi a)
\end{aligned}$$

$$= \hat{D}_a M_k \hat{f}(\xi). \quad (1.17)$$

This provides us with useful properties of the dilation and translation operators under the Fourier transform. It is also important for us to establish the following well known propositions. They will allow us to take advantage of the dilation and translation notation.

Proposition 1.1. *For any matrix $c \in GL_n(\mathbb{R})$, the inverse dilation operator is $D_c^{-1} = D_{c^{-1}}$.*

Proof. This is obvious since

$$D_{c^{-1}}[D_c f](x) = |\det(c)|^{\frac{1}{2}} [D_c f](cx) = |\det(c)|^{-\frac{1}{2}} |\det(c)|^{\frac{1}{2}} f(c^{-1}cx) = f(x).$$

So $D_c^{-1} = D_{c^{-1}}$. \square

Proposition 1.2. *For any matrix $c \in GL_n(\mathbb{R})$, $(D_c^j)^* = D_c^{-j}$.*

Proof. This is a straightforward calculation involving the inner product and a change of variables. Let $f, g \in L^2(\mathbb{R}^n)$.

$$\begin{aligned} \langle f, D_c^j g \rangle &= \int_{\mathbb{R}^n} f(x) \overline{[D_c^j g](x)} dx \\ &= \int_{\mathbb{R}^n} f(x) |\det(c)|^{-\frac{j}{2}} \overline{g(c^{-j}x)} dx \\ &= |\det(c)|^{-\frac{j}{2}} \int_{\mathbb{R}^n} f(c^j y) \overline{g(y)} |\det(c)|^j dy \\ &= \int_{\mathbb{R}^n} |\det(c)|^{\frac{j}{2}} f(c^j y) \overline{g(y)} dy \\ &= \int_{\mathbb{R}^n} [D_c^{-j} f](y) \overline{g(y)} dy \\ &= \langle D_c^{-j} f, g \rangle \end{aligned}$$

Hence, $(D_c^j)^* = D_c^{-j}$. \square

Proposition 1.3. For $k \in \Gamma = c\mathbb{Z}^n$, $T_k^{-1} = T_k^* = T_{-k}$.

Proof. These are also straightforward observations. First of all

$$T_{-k}T_k f(x) = T_k f(x+k) = f(x+k-k) = f(x)$$

so $T_k^{-1} = T_{-k}$.

Now let $f, g \in L^2(\mathbb{R}^n)$.

$$\begin{aligned} \langle f, T_k g \rangle &= \int_{\mathbb{R}^n} f(x) \overline{T_k g(x)} dx \\ &= \int_{\mathbb{R}^n} f(x) \overline{g(x-k)} dx \\ &= \int_{\mathbb{R}^n} f(y+k) \overline{g(y)} dy \\ &= \int_{\mathbb{R}^n} T_{-k} f(y) \overline{g(y)} dy \\ &= \langle T_{-k} f, g \rangle. \end{aligned}$$

Hence, $T_k^* = T_{-k}$. Thus, $T_k^{-1} = T_k^* = T_{-k}$. \square

1.2 Group Actions on \mathbb{R}^n

This section describes how a group acts on \mathbb{R}^n and establishes the notation and terminology we use when discussing the groups. We are mostly interested in how the group B will affect \mathbb{R}^n . We primarily concern ourselves with the case when $B \subset GL_n(\mathbb{R})$ is a finite group of invertible matrices. First of all, we establish some nomenclature that is not necessarily standard.

Regarding notation, writing $A \cap B = \emptyset$ is meant as essentially disjoint. That is, this equation does not preclude the sets A and B from sharing a set of measure zero. This is often referred to as being disjoint in the sense of measure, i.e. they have an

intersection of measure 0. In the same spirit, when we refer to two sets as "disjoint" we still imply that they may share a set of measure zero.

We make the following definitions to assist in distinguishing between how we are interpreting a group action.

Definition 1.9. Let B be a finite subgroup of $GL_n(\mathbb{R})$. The *fundamental region* for B is a set, $F \subset \hat{\mathbb{R}}^n$, such that

- (i) $\bigcup_{b \in B} Fb = \hat{\mathbb{R}}^n$ and
- (ii) for $b_1 \neq b_2 \in B$, $Fb_1 \cap Fb_2 = \emptyset$.

Figure 1.1 shows a fundamental region for the group generated by the reflections through the hyperplanes $y = x$ and $y = 0$. We let r_1 represent the reflection through the line $y = x$ and r_2 represent the reflection through $y = 0$. We will use the standard notation for generating a group, namely $B = \langle r_1, r_2 \rangle$.

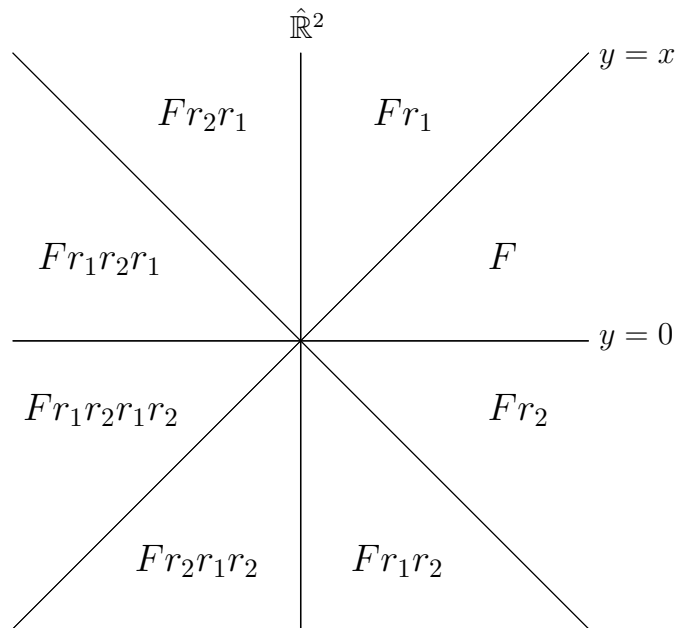


Figure 1.1: F is a fundamental region for $B = \langle r_1, r_2 \rangle$, the reflections through $y = x$ and $y = 0$.

Definition 1.10. Let G be a group acting on a set Ω . Then R is a *G-tiling set* for Ω if

- (i) $\bigcup_{g \in G} Rg = \Omega$ and
- (ii) for $g_1 \neq g_2 \in G$, $Rg_1 \cap Rg_2 = \emptyset$.

Figure 1.2 shows a B -tiling set for the two dimensional example where $B = \langle r_1, r_2 \rangle$, the group generated by reflections through the line $y = x$ and $y = 0$. The triangle, R , is a B -tiling set for the square, S .

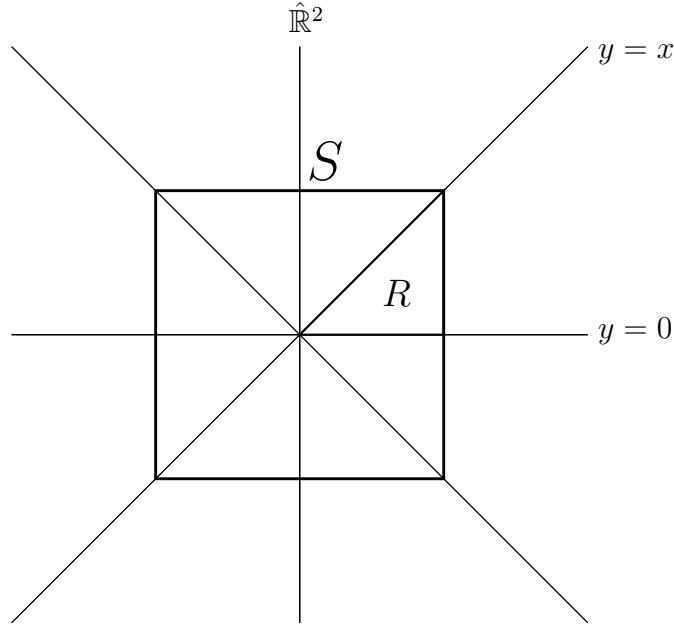


Figure 1.2: The triangle R is a B -tiling set for the square S , where $B = \langle r_1, r_2 \rangle$, the reflections through $y = x$ and $y = 0$.

These two definitions have very minor differences. The first definition refers specifically to the subset of $\hat{\mathbb{R}}^n$ that, when acted upon by B , will generate all of $\hat{\mathbb{R}}^n$ without any measurable overlap. Obviously, if F is the fundamental region for B , then F is a B -tiling set for $\hat{\mathbb{R}}^n$. The first definition is also restricted to finite subgroups of the invertible matrices. This is to avoid calling a lattice tiling set a fundamental region. Many authors use this terminology, but here we want to distinguish the set F related to the group used in the composite dilations. For example, the unit interval $[0, 1]$ is a \mathbb{Z} -tiling set for \mathbb{R} . The restriction to finite subgroups of the invertible matrices prevents the minor confusion of calling $[0, 1]$ a fundamental region for \mathbb{Z} .

Much of chapter 2 deals with groups with a certain property for their fundamental

domains. The chapter is devoted to proving the existence of minimally supported frequency wavelets for a group B and a lattice Γ . Whenever a group, B , has the property that its fundamental region, F , is bounded by hyperplanes through the origin, we are able to prove the existence of minimally supported composite dilation wavelets for every lattice Γ (see Theorem 2.10). A *hyperplane* is a codimension 1 linear subspace. We simply want every hyperplane bounding the fundamental region to pass through the origin. When this occurs, F is an n -dimensional cone-like region with the origin as its vertex.

A large, well-known, classified family of finite groups with fundamental regions bounded by hyperplanes is the family of Coxeter groups [7]. Coxeter groups, also known as reflection groups, are generated by reflections through the hyperplanes bounding their fundamental region. In [7], the hyperplanes bounding the fundamental region are referred to as mirrors. Let us examine our notation and definitions 1.9 and 1.10 with an example.

Example 1.1. Let B be the symmetries of the square acting on $\hat{\mathbb{R}}^2$. Then B is a Coxeter group generated by two reflections, r_1 , the reflection through $y = x$ and, r_2 , the reflection through the line $y = 0$. These reflections are determined by the lines $y = 0$ and $y = x$. These lines are hyperplanes through the origin in $\hat{\mathbb{R}}^2$. The reflections are:

$$r_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$r_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let $F = \{(x, y) : x \geq y\} \cap \{(x, y) : y \geq 0\}$. Then F is bounded by the hyperplanes $H_1 = \{(x, y) : y = x\}$ and $H_2 = \{(x, y) : y = 0\}$. If we let $B = \langle r_1, r_2 \rangle$ act on F , we

see that F is a fundamental region for B .

Now let S be the unit square centered at the origin, $S = \{(x, y) : -\frac{1}{2} \leq x, y \leq \frac{1}{2}\}$. Let R be the triangle with vertices $(0, 0)$, $(\frac{1}{2}, 0)$, and $(\frac{1}{2}, \frac{1}{2})$. Then R is also bound by H_1 and H_2 . When B acts on R , we see that R is a B -tiling set for S .

Coxeter groups have been classified and exist in every dimension. A Coxeter group acting on \mathbb{R}^n will be generated by at most n reflections. This, combined with the significant literature available on Coxeter groups, suggests them as excellent candidates for use when constructing composite dilation wavelets. Another reasonably attractive option, especially in lower dimensions, is a rotation group. Rotation groups are subgroups of Coxeter groups. These may be the best choices for finite groups with composite dilation wavelets. As noted in chapter 2, Coxeter groups and rotation groups have fundamental regions bounded by hyperplanes through the origin. For more information on Coxeter groups, the reader should see Grove and Benson's Finite Reflection Groups [7] or go directly to the source in [3] and [4].

Composite dilation wavelets require the identification of an associated full rank lattice.

Definition 1.11. Γ is a *Full Rank Lattice* if $\Gamma = c\mathbb{Z}^n$ where c is an invertible matrix, $c \in GL_n(\mathbb{R})$.

Definition 1.12. Let $\Gamma = c\mathbb{Z}^n$ be a full rank lattice. A *Lattice Basis* is a set $\{\gamma_i\}_{i=1}^n$ such that every element $\gamma \in \Gamma$ can be written as an integer combination of the elements:

$$\gamma = \sum_{i=1}^n m_i \gamma_i \text{ where } m_i \in \mathbb{Z}.$$

If c_i is the i th column of the matrix c , then the set $\{c_i\}_{i=1}^n$ is a basis for Γ . When we move between the time and frequency domains, we must deal with the appropriate corresponding lattices.

Definition 1.13. Suppose $\Gamma = c\mathbb{Z}^n$ is a full rank lattice in \mathbb{R}^n . The *dual lattice* to Γ is the lattice $\Gamma^* = \hat{\mathbb{Z}}^n c^{-1}$, a full rank lattice in $\hat{\mathbb{R}}^n$. The rows of the matrix c^{-1} form a basis for the dual lattice Γ^* .

The notion of a dual lattice is useful when defining bases on subspaces of $L^2(\mathbb{R}^n)$. For minimally supported frequency wavelets, we start with the characteristic function of a set, say R . When R is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$, we can generate an orthonormal basis for $L^2(R)$ by using exponential functions associated with the lattice Γ . The duality of the lattices makes this system orthonormal. This will be used extensively in chapter 2.

Finally, we define a group acting on a lattice (or a subset of a lattice) by $B(\Gamma) = \bigcup_{b \in B} \Gamma b$. We will frequently want to choose a group and a lattice with some symmetry. For example, we will often choose Γ so that $B(\Gamma) = \Gamma$ or choose an expanding matrix a so that $a(\Gamma) \subset \Gamma$.

1.3 Accuracy of Refinable Functions

In chapter 3 we discuss composite dilation wavelets with some special, advantageous properties. We look for composite dilation wavelets with compact support in the time domain. In [13], examples are presented of composite dilation wavelets which are characteristic functions of sets in the time domain. It would be very useful to have composite dilation wavelets with greater smoothness than the characteristic functions. Smooth, compactly supported wavelets provide localization in both the time and frequency domains. In classical wavelet literature, Daubechies compactly supported wavelets [5] have arbitrary smoothness. There has also been reasonably significant advances in finding arbitrarily smooth, non separable wavelets in two dimensions, for example in [1] and [12].

In [1] Belogay and Wang use the notion of accuracy to measure the smoothness of

a function. For affine systems with multiple generating functions, Cabrelli, Heil, and Molter have described the necessary conditions to obtain arbitrary levels of accuracy [2]. We adopt this approach in our hunt for composite dilation wavelets with accuracy greater than the known Haar-type wavelets discussed in [13]. In order to fully utilize this approach, we must understand much of the notation, ideas, and results from [2]. We continue to use full rank lattices, $\Gamma = c\mathbb{Z}^n$.

In this section and in chapter 3, the notation of dimension changes from n -dimensional to d -dimensional. Also, here and in chapter 3, the meaning of A has changed. Above and in chapter 2, A was a group generated by an expanding matrix a . Here we use A to denote the expanding matrix, not the group. This allows us to utilize the generalized matrix notation and operator notation in [2]. These minor inconveniences are worthwhile to use the notation from [2].

Definition 1.14. A *dilation matrix associated with* Γ is a $d \times d$ matrix A such that

- (i) $A(\Gamma) \subset \Gamma$; and
- (ii) A is expanding, i.e. all eigenvalues satisfy $|\lambda_k(A)| > 1$.

Suppose we have a set of functions $f_1, \dots, f_r \in L^2(\mathbb{R}^n)$. Suppose they generate an affine system

$$\mathcal{A}_{A\Gamma} = \{D_A^j T_k f_i : j \in \mathbb{Z}, k \in \Gamma, i = 1, \dots, r\}.$$

In order to have an affine system that may lead to an MRA, we want to ensure that we can write the dilations of our generating functions as linear combinations of the translates of the generating functions. Moreover, we want to find functions with compact support. In [6], lemma 5 states that a function satisfying the refinement equation for a finite subset of the lattice must have compact support. Therefore, we further limit our search to finite subsets of the associated lattice, $\Lambda \subset \Gamma$, Λ finite.

Definition 1.15. The functions $f_1, \dots, f_r \in L^2(\mathbb{R}^d)$ are *refinable* with respect to A

and Γ if there exists $\Lambda \subset \Gamma$ (Λ finite) and a vector-valued function $F = (f_1, \dots, f_r)^t : \mathbb{R}^d \rightarrow \mathbb{R}^r$ satisfying the *refinement equation*

$$F(x) = \sum_{k \in \Lambda} c^k F(Ax - k) \quad (1.18)$$

for some $r \times r$ matrices c^k .

We will refer to both F and $\{f_1, \dots, f_r\}$ as *refinable*. Refinable functions are exactly those functions whose affine systems will generate an MRA.

It is valuable to have certain levels of smoothness associated with our refinable functions. One method of measuring the smoothness of an affine system is to determine the *accuracy* of the system. We define accuracy in terms of the shift invariant space generated by F .

Definition 1.16. The *shift invariant space generated by F and Γ* is the set of all linear combinations of the lattice translates of the functions f_1, \dots, f_r :

$$S(F) = \left\{ \sum_{k \in \Gamma} \beta_k T_k F : \beta_k \in \mathbb{R}^{1 \times r} \right\} = \left\{ \sum_{k \in \Gamma} \sum_{i=1}^r \beta_{k,i} T_k f_i : \beta_{k,i} \in \mathbb{R} \right\}. \quad (1.19)$$

Since we are searching for compactly supported functions F (or f_1, \dots, f_r), the series in (1.19) are well-defined for all choices of b_k (or $b_{k,i}$). We interpret equality between two functions f and g to mean $f = g$ almost everywhere. The fundamental property of the functions F that is studied in [2] and used to measure smoothness in [1] is the property of *accuracy*. Here we define the accuracy of F , which obviously can also be interpreted as the accuracy of the collection $\{f_1, \dots, f_r\}$.

Definition 1.17. The function $F : \mathbb{R}^d \rightarrow \mathbb{R}^r$ has *accuracy p* if p is the largest integer such that all multivariate polynomials $q(x) = q(x_1, \dots, x_d)$ with $\deg(q) < p$ lie in the shift invariant space $S(F)$.

Composite dilation wavelets and the associated composite dilation scaling functions are elements of $L^2(\mathbb{R}^d)$ while no multivariate polynomial lies in $L^2(\mathbb{R}^d)$. To generate an MRA with a scaling function Φ , we measure the accuracy of Φ in $S(\Phi)$ and take $V_0 = S(\Phi) \cap L^2(\mathbb{R}^d)$. When Φ is a refinable function, we produce the MRA as usual by defining $V_j = D_A^{-j} V_0$. An MRA composite dilation wavelet, Ψ , has accuracy p whenever the scaling function Φ has accuracy p since $\Psi(x) = \sum_{k \in \Gamma} \beta_k D_A T_k \Phi(x)$ for some collection $\{\beta_k\} \subset \mathbb{R}^{1 \times r}$.

It is important to understand the relationship between accuracy and smoothness. Belogay and Wang use accuracy to measure smoothness and describe an asymptotic relationship between accuracy and smoothness [1]. Daubechies employed vanishing moments to measure smoothness when describing her compactly supported wavelets [5]. Since the reader is likely more familiar with the Daubechies wavelets, it is helpful to interpret accuracy in terms of vanishing moments.

Suppose Φ is a compactly supported, refinable function as above. Suppose $d = 1$ so $\Phi : \mathbb{R} \rightarrow \mathbb{R}^r$. Now suppose Φ has accuracy p . Then the monomial x^s lies in the shift invariant space $S(\Phi)$ for all $s < p$. Since the restriction of a monomial to a compact set is square integrable, then for every $M \in \mathbb{N}$, $x^s \chi_{[-M, M]} \in S(\Phi) \cap L^2(\mathbb{R}^d)$. Therefore, there exists a sequence $\{b_k : k \in \Gamma\}$ such that

$$x^s \chi_{[-M, M]}(x) = \sum_{k \in \Gamma} b_k T_k \Phi(x)$$

for all $s < p$.

Suppose Φ was a scaling function for an MRA, $S(\Phi) \cap L^2(\mathbb{R}^d) = V_0$. Since Φ was refinable, $V_j = D_A^{-j} V_0$ will establish an MRA. If Ψ is the associated wavelet, $S(\Psi) \cap L^2(\mathbb{R}^d) = W_0 \subset V_1$.

Now $\Psi : \mathbb{R} \rightarrow \mathbb{R}^r$ is a compactly supported, refinable function such that $V_0 \perp W_0$

$(S(\Phi) \perp S(\Psi))$. Then, for every $M \in \mathbb{N}$,

$$\begin{aligned} \int_{-M}^M x^p \Psi(x) dx &= \int_{-M}^M \sum_{k \in \Gamma} b_k T_k \Phi(x) \Psi(x) dx \\ &= \sum_{k \in \Gamma} b_k \int_{-M}^M T_k \Phi(x) \Psi(x) dx \\ &= 0 \end{aligned}$$

since $S(\Phi) \perp S(\Psi)$. That is, Ψ has $p - 1$ vanishing moments on every compact subset of \mathbb{R} .

Therefore, an MRA wavelet system with accuracy p has $p - 1$ vanishing moments on every compact subset of \mathbb{R} . Higher dimensional analogues are true as well. The interpretation of MRA, composite dilation wavelets used for chapter 3 sees $\Psi = (D_{b_1} \psi, \dots, D_{b_r} \psi)^t$ where B is a group of order r . This allows us to use the above the notation and interpretation of smoothness via accuracy for composite dilation wavelets.

To obtain the necessary conditions for the existence of such refinable functions, Cabrelli, Heil and Molter use a generalized matrix notation and carry this notation over to operators. We will use theorems 3.4 and 3.6 from [2]. In order to do so, we will establish some of their notation in the following discussion. For a complete understanding of the notation and for proofs of the theorems see [2].

We will use the following combined version of Theorems 3.4 and 3.6 from [2].

Theorem (Cabrelli, Heil, Molter, [2]): Assume $\Phi(x) : \mathbb{R}^d \rightarrow \mathbb{C}^r$ is a refinable, integrable, compactly supported function with independent \mathbb{Z}^d -translates. Then Φ has accuracy p if and only if there exists a collection of row vectors $\{v_{[s]} \in \mathbb{C}^{1 \times r} : 0 \leq s < p\}$ such that

(i) $v_{[0]} \hat{\Phi}(0) \neq 0$, and

(i) $v_{[s]} = \sum_{k \in \Gamma_i} \sum_{t=0}^s Q_{[s,t]} A_{[t]} v_{[t]} c^k$

where A , the expanding matrix, and $\{c^k : k \in \Gamma\}$, the collection of $r \times r$ matrices, are from the refinement equation.

In order to decipher this theorem, we must understand the notation. We are already aware of the meaning of a refinable, integrable, compactly supported function. We are also aware of the meaning of accuracy. Therefore, we must define *independent translates*, the collection of row vectors $\{v_{[s]} \in \mathbb{C}^{1 \times r} : 0 \leq s < p\}$, $Q_{[s,t]}$, and $A_{[t]}$.

Definition 1.18. $F : \mathbb{R}^d \rightarrow \mathbb{R}^r$ has *independent Γ -translates* if for every choice of row vectors $b_k \in \mathbb{R}^{1 \times r}$,

$$\sum_{k \in \Gamma} b_k T_k F(x) = \sum_{k \in \Gamma} b_k F(x - k) = 0 \Leftrightarrow b_k = 0 \text{ for every } k.$$

Cabrelli, Heil, and Molter introduce a generalized matrix notation where the entries of a matrix are not limited to scalars, but may themselves be matrices. First we introduce a multi-index, $\alpha = (\alpha_1, \dots, \alpha_d)$ with α_i a nonnegative integer. The degree of α is $|\alpha| = \alpha_1 + \dots + \alpha_d$. The number of multi-indices of degree s is $d_s = \binom{s+d-1}{d-1}$. Write $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for all $i = 1, \dots, d$. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we have $x^\alpha = (x_1^{\alpha_1}, \dots, x_d^{\alpha_d})$.

Suppose we have a collection of $1 \times r$ row vectors

$$\{v_\alpha = (v_{\alpha,1}, \dots, v_{\alpha,r}) \in \mathbb{C}^{1 \times r} : 0 \leq |\alpha| < p\}. \quad (1.20)$$

We group the row vectors by degree: $v_{[s]} = [v_\alpha]_{|\alpha|=s}$. This is the generalized matrix notation where $v_{[s]}$ is a $d_s \times 1$ matrix whose entries are $1 \times r$ row vectors. Then the collection (1.20) can be written as

$$\{v_{[s]} \in \mathbb{C}^{1 \times r} : 0 \leq s < p\}. \quad (1.21)$$

The operators $Q_{[s,t]}$ and $A_{[t]}$ are necessary to define translation and dilation on multidimensional monomials. Since $x^\alpha = (x_1^{\alpha_1}, \dots, x_d^{\alpha_d})$, we have a vector valued function $X_{[s]} : \mathbb{R}^d \rightarrow \mathbb{R}^{d_s}$ defined by

$$X_{[s]}(x) = [x^\alpha]_{|\alpha|=s}. \quad (1.22)$$

$Q_{[s,t]}$ is the matrix of polynomials which arise from applying a translation operator to $X_{[s]}$.

Definition 1.19. $Q_{[s,t]}$ is the matrix of polynomials satisfying the equation

$$X_{[s]}(x - k) = \sum_{t=0}^s Q_{[s,t]}(k) X_{[s]}(x).$$

Similarly, $A_{[s]}$ is the $d_s \times d_s$ matrix associated with applying a dilation by A to the vector valued function $X_{[s]}$.

Definition 1.20. $A_{[s]}$ is the $d_s \times d_s$ matrix satisfying the equation

$$X_{[s]}(Ax) = A_{[s]} X_{[s]}(x).$$

In [2], $Q_{[s,t]}$ and $A_{[s]}$ are developed explicitly in section 2.4. Obviously, when dealing with one dimensional functions (i.e. $d = 1$) the notation becomes much simpler. Since there is only one index with degree s , then $v_{[s]} = v_s$ and $A_{[s]} = A^s$. Also, $Q_{[s,t]}$ becomes the coefficients of $(x - k)^s$. Thus, for a translation by k ,

$$Q_{[s,t]}(k) = \sum_{t=0}^s \binom{s}{t} (-k)^{s-t}. \quad (1.23)$$

In chapter 3 we use the ideas from [2] to determine the necessary conditions for

a composite dilation system to have accuracy p . Interpreting a composite dilation system as a set of r functions where r is the order of the composite dilation group B , $r = |B|$, we may apply the results from [2]. Moreover, the nature of the composite system allows us to reduce the above the situation to equations on the entries of the coefficient matrices, c^k , defined above.

Chapter 2

Existence of Minimally Supported Frequency Composite Dilation Wavelets

This chapter provides sufficient conditions for the existence of minimally supported frequency composite dilation wavelets. In section 2.1, we establish two admissibility conditions that are sufficient to produce minimally supported frequency composite dilation wavelets. Section 2.2 provides a large family of groups that will admit an arbitrary lattice. Section 2.3 proves that, in every dimension, $2(I_n)$ satisfies the admissibility condition for expanding matrices. As a result, this section ends by proving that MSF, MRA composite dilation wavelets exist in every dimension. It is advantageous to minimize the number of generating functions in the wavelet systems. Families of singly-generated MSF composite dilation wavelets are developed in section 2.4. The chapter concludes with section 2.5 which discusses similar results when the composite dilation wavelets generate Parseval frames rather than orthonormal bases.

2.1 Admissibility Conditions

This section develops sufficient conditions on composite dilation groups, full rank lattices, and expanding matrices that admit minimally supported frequency composite dilation wavelets. We begin by defining two admissibility conditions. The first condition describes when a full rank lattice and the group used for composite dilations will provide the appropriate sets to support the Fourier transform of the scaling function. The second admissibility condition determines when an expanding matrix will provide appropriate sets to support the Fourier transform of the wavelet generating functions. These two admissibility conditions are then shown to be sufficient for the existence of MSF, MRA composite dilation wavelets. We begin with a few of definitions.

Definition 2.1. A set S is *starlike with respect to x* if, for every $y \in S$, the convex combination of y and x lies in S : $\{tx + (1 - t)y : t \in [0, 1]\} \subset S$.

Definition 2.2. A *Starlike Neighborhood of x* is a set S such that S is starlike with respect to x and S contains an open neighborhood of x .

Definition 2.3. A full rank lattice $\Gamma = c\mathbb{Z}^n$ ($c \in GL_n(\mathbb{R})$) is *B -Admissible* if there exist a group B and a measurable set $R \subset \hat{\mathbb{R}}^n$ such that:

- (i) R is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$ and
- (ii) there exists a starlike neighborhood of 0, S , such that R is a B -tiling set for S .

To better understand definitions 2.2 and 2.3, we look at a simple example in two dimensions. Let B be the group of symmetries of the square generated by the reflection through $y = x$ and $y = 0$. Let Γ be the full rank lattice, $\Gamma = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mathbb{Z}^2$. Then figure 2.1 shows a set R that satisfies (i) and (ii) from definition 2.3. The set S in figure 2.1 is a starlike neighborhood of the origin.

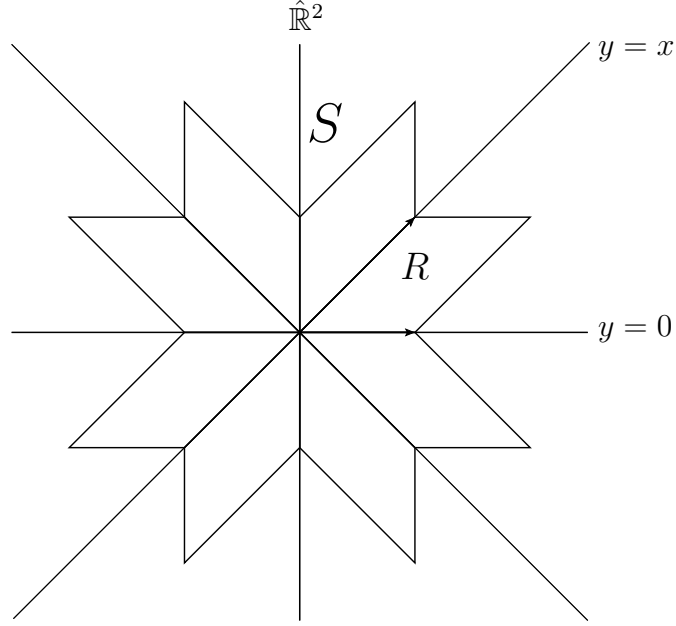


Figure 2.1: R is a Γ^* -tiling set for $\hat{\mathbb{R}}^2$ and a B -tiling set for S .

Definition 2.4. Let Γ be B -admissible and let S be a starlike neighborhood of 0 satisfying (ii) from definition 2.3. A matrix $a \in GL_n(\mathbb{R})$ is (B, Γ) -admissible if a is expanding with $S \subset Sa$ and there exist disjoint sets $R_1, \dots, R_L \subset Sa \setminus S$ such that:

- (i) for all $l = 1 \dots L$, R_l is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$ and
- (ii) $\bigcup_{l=1}^L R_l$ is a B -tiling set for $Sa \setminus S$.

Continuing our two dimensional example, we have B , the group of symmetries of the square, and $\Gamma = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mathbb{Z}^2$. Then $a = 2(I_2)$ is (B, Γ) -admissible as shown in figure 2.2. We see that R_1, R_2 , and R_3 are Γ^* -tiling sets for $\hat{\mathbb{R}}^2$ and that $R_1 \cup R_2 \cup R_3$ is a B -tiling set for $Sa \setminus S$. From this figure we see how the MSF, MRA wavelets are defined. The scaling function φ is defined by $\hat{\varphi} = \chi_R$, while the wavelets, ψ^l , are defined by $\hat{\psi}^l = \chi_{R_l}$ for $l = 1, 2, 3$.

It only makes sense to define these two admissibility conditions if they are sufficient for constructing composite dilation MSF wavelets. Indeed they are as the following theorem establishes.

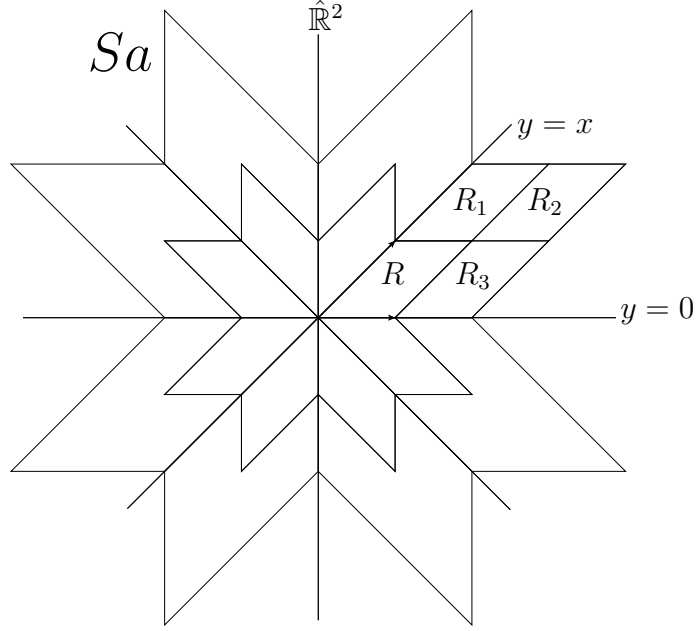


Figure 2.2: R_1, R_2, R_3 are Γ^* -tiling sets for $\hat{\mathbb{R}}^2$ and $R_1 \cup R_2 \cup R_3$ is a B -tiling set for $Sa \setminus S$.

Theorem 2.1. *Let B be a group and $\Gamma = c\mathbb{Z}^n$ a B -admissible lattice. Let $R \subset \hat{\mathbb{R}}^n$ be a set satisfying (i) and (ii) from definition 2.3. Let $a \in GL_n(\mathbb{R})$ be a (B, Γ) -admissible matrix with sets R_1, \dots, R_L satisfying (i) and (ii) from definition 2.4. Let $\psi^1, \dots, \psi^L \in L^2(\mathbb{R}^n)$ be functions such that $\hat{\psi}^l = |\det(c)|^{\frac{1}{2}} \chi_{R_l}$ for $1 \leq l \leq L$. Then*

$$\Psi = \{D_a^j D_b T_k \psi^l : j \in \mathbb{Z}, b \in B, k \in \Gamma, 1 \leq l \leq L\}$$

is an orthonormal $AB\Gamma$ -MRA composite dilation wavelet.

Proof. R is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$. Then $\|\chi_R\|_2 = |\det(c)|^{-\frac{1}{2}}$. Also, since R is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$, the set

$$\{e_k(\xi) : k \in \Gamma\} := \left\{ |\det(c)|^{\frac{1}{2}} M_k \chi_R(\xi) : k \in \Gamma \right\} := \left\{ |\det(c)|^{\frac{1}{2}} e^{-2\pi i \xi k} \chi_R(\xi) : k \in \Gamma \right\} \quad (2.1)$$

is an orthonormal basis for $L^2(R)$. Scaling the functions $M_k \chi_R$ by $|\det(c)|^{\frac{1}{2}}$ provides the necessary normalization.

Now $\hat{D}_{b^{-1}} : L^2(R) \rightarrow L^2(Rb)$ is a unitary operator. Thus, $\{\hat{D}_{b^{-1}}e_k(\xi) : k \in \Gamma\}$ is an orthonormal basis for $L^2(Rb)$. Let $S = \bigcup_{b \in B} Rb$. Then $L^2(S) = L^2(\bigcup_{b \in B} Rb) = \bigoplus_{b \in B} L^2(Rb)$. Therefore,

$$\bigoplus_{b \in B} \{\hat{D}_{b^{-1}}e_k(\xi) : k \in \Gamma\} = \{\hat{D}_{b^{-1}}e_k(\xi) : b \in B, k \in \Gamma\} \quad (2.2)$$

is an orthonormal basis for $L^2(S)$. We now take the inverse Fourier transform of this basis to obtain a basis for $\check{L}^2(S)$.

$$\begin{aligned} (\hat{D}_{b^{-1}}e_k)^\vee(x) &= \int_{\mathbb{R}^n} \hat{D}_{b^{-1}}e_k(\xi) e^{2\pi i \xi x} d\xi \\ &= \int_{\mathbb{R}^n} e_k(\xi b^{-1}) e^{2\pi i \xi x} d\xi \\ &= \int_{\mathbb{R}^n} e_k(\omega) e^{2\pi i \omega b x} d\omega \\ &= |\det(c)|^{\frac{1}{2}} \int_{\mathbb{R}^n} e^{-2\pi i \omega k} \chi_R(\omega) e^{2\pi i \omega b x} d\omega \\ &= |\det(c)|^{\frac{1}{2}} \int_{\mathbb{R}^n} \chi_R(\omega) e^{2\pi i \omega (bx - k)} d\omega \\ &= |\det(c)|^{\frac{1}{2}} \check{\chi}_R(bx - k) \\ &= |\det(c)|^{\frac{1}{2}} D_{b^{-1}} T_k \check{\chi}_R(x) \end{aligned}$$

So we have an orthonormal basis for $\check{L}^2(S)$:

$$\left\{ |\det(c)|^{\frac{1}{2}} D_b T_k \check{\chi}_R(x) : b \in B, k \in \Gamma \right\}. \quad (2.3)$$

Let $V_0 = \check{L}^2(S)$. Define $\varphi(x) = |\det(c)|^{\frac{1}{2}} \check{\chi}_R(x)$. Then $\varphi(x)$ satisfies (M5).

Since a is (B, Γ) -admissible, then $S \subset Sa$. Therefore, for all $j \in \mathbb{Z}$, we define

$V_j = D_a^{-j}V_0 = \check{L}^2(Sa^j)$. From this, we see that $V_0 \subset V_1$:

$$L^2(Sa) = L^2((Sa \setminus S) \cup S) = L^2(Sa \setminus S) \oplus L^2(S)$$

Then $\check{L}^2(S) \subset \check{L}^2(Sa)$, which is $V_0 \subset V_1$. Therefore, for all $j \in \mathbb{Z}$,

$$V_j = D_a^{-j}V_0 \subset D_a^{-j}V_1 = D_a^{-j}D_a^{-1}V_0 = D_a^{-(j+1)}V_0 = V_{j+1}.$$

Thus, (M1) is satisfied.

(M2) is satisfied by the way we defined the sets V_j : if $f \in V_j = D_a^{-j}V_0$, then $D_a f \in D_a V_j = D_a^{-(j+1)}V_0 = V_{j+1}$.

Since $Sa^j \subset Sa^{j+1}$ for all $j \in \mathbb{Z}$, $\bigcup_{j=-N}^N Sa^j = Sa^N$. Then, since a is expanding,

$$\lim_{N \rightarrow \infty} \bigcup_{j=-N}^N Sa^j = \lim_{N \rightarrow \infty} Sa^N = \hat{\mathbb{R}}^n.$$

So we have

$$\overline{\bigcup_{j \in \mathbb{Z}} L^2(Sa^j)} = \lim_{N \rightarrow \infty} \bigcup_{j=-N}^N L^2(Sa^j) = L^2(\lim_{N \rightarrow \infty} Sa^N) = L^2(\hat{\mathbb{R}}^n).$$

Therefore, (M3) is satisfied by

$$\overline{\bigcup_{j \in \mathbb{Z}} V_j} = \overline{\bigcup_{j \in \mathbb{Z}} \check{L}^2(Sa^j)} = \check{L}^2(\hat{\mathbb{R}}^n) = L^2(\mathbb{R}^n). \quad (2.4)$$

In a similar fashion, we establish (M4). Since $Sa^{-(j+1)} \subset Sa^{-j}$ then $\bigcap_{j=-N}^N Sa^j = Sa^{-N}$. Therefore

$$\bigcap_{j \in \mathbb{Z}} Sa^j = \lim_{N \rightarrow \infty} \bigcap_{j=-N}^N Sa^j = \lim_{N \rightarrow \infty} Sa^{-N} = 0.$$

Hence,

$$\begin{aligned}
\bigcap_{j \in \mathbb{Z}} V_j &= \bigcap_{j \in \mathbb{Z}} \check{L}^2(Sa^j) \\
&= \check{L}^2\left(\bigcap_{j \in \mathbb{Z}} Sa^j\right) \\
&= \check{L}^2(0) = 0
\end{aligned}$$

This establishes an MRA. In order to have an orthonormal wavelet system, we must find the orthogonal complement to V_0 in V_1 . By the standard MRA wavelet construction, if we find an orthonormal basis for $W_0 = V_1 \cap V_0^\perp$, then we have a wavelet system. Since $V_0 = \check{L}^2(S)$ and $V_1 = \check{L}^2(Sa)$, then $V_1 \cap V_0^\perp = \check{L}^2(Sa) \cap \check{L}^2(S)^\perp = \check{L}^2(Sa \setminus S)$. So define $W_0 = \check{L}^2(Sa \setminus S)$.

Using arguments similar to those leading to (2.1) and (2.2), we can find an orthonormal basis for $L^2(R_l b)$ for each $b \in B$:

$$\left\{ |\det(c)|^{\frac{1}{2}} e^{-2\pi i \xi b^{-1} k} \chi_{R_l}(\xi b^{-1}) : k \in \Gamma \right\}.$$

Therefore, for each $b \in B$ we have an orthonormal basis for $\check{L}^2(R_l b)$ for all $1 \leq l \leq L$:

$$\left\{ |\det(c)|^{\frac{1}{2}} D_b T_k \check{\chi}_{R_l}(x) : k \in \Gamma \right\}.$$

For each $1 \leq l \leq L$, let $\psi_l \in L^2(\mathbb{R}^n)$ be a function such that $\hat{\psi}_l(\xi) = |\det(c)|^{\frac{1}{2}} \chi_{R_l}(\xi)$. Then $\psi^l(x) = |\det(c)|^{\frac{1}{2}} \check{\chi}_{R_l}$.

By assumption, $\bigcup_{l=1}^L R_l$ is a B -tiling set for $Sa \setminus S$. Thus

$$W_0 = \check{L}^2(Sa \setminus S) = \check{L}^2\left(\bigcup_{b \in B} \bigcup_{l=1}^L R_l b\right)$$

$$\begin{aligned}
&= \bigoplus_{b \in B} \bigoplus_{l=1}^L \tilde{L}^2(R_l b) \\
&= \bigoplus_{b \in B} \overline{\text{span}} \left\{ |\det(c)|^{\frac{1}{2}} D_b T_k \tilde{\chi}_{R_l}(x) : k \in \Gamma, 1 \leq l \leq L \right\} \\
&= \overline{\text{span}} \left\{ |\det(c)|^{\frac{1}{2}} D_b T_k \tilde{\chi}_{R_l}(x) : b \in B, k \in \Gamma, 1 \leq l \leq L \right\} \\
&= \overline{\text{span}} \left\{ D_b T_k \psi^l(x) : b \in B, k \in \Gamma, 1 \leq l \leq L \right\}.
\end{aligned}$$

Defining $W_j = D_a^{-j} W_0$ provides a sequence of spaces such that $V_j \oplus W_j = V_{j+1}$ and $V_j \cap W_j = \emptyset$. Then, as in the standard MRA argument,

$$V_{j+1} = V_j \oplus W_j = V_{j-1} \oplus W_{j-1} \oplus W_j = \cdots = \bigoplus_{l=-\infty}^j W_l. \quad (2.5)$$

Therefore, using (2.4) and (2.5), we have

$$\bigoplus_{j \in \mathbb{Z}} W_j = L^2(\mathbb{R}^n). \quad (2.6)$$

Finally

$$\begin{aligned}
\bigoplus_{j \in \mathbb{Z}} W_j &= \bigoplus_{j \in \mathbb{Z}} D_a^{-j} W_0 \\
&= \bigoplus_{j \in \mathbb{Z}} \overline{\text{span}} \left\{ D_a^{-j} D_b T_k \psi^l(x) : b \in B, k \in \Gamma, 1 \leq l \leq L \right\} \\
&= \overline{\text{span}} \left\{ D_a^{-j} D_b T_k \psi^l(x) : j \in \mathbb{Z}, b \in B, k \in \Gamma, 1 \leq l \leq L \right\} \\
&= \overline{\text{span}} \left\{ D_a^j D_b T_k \psi^l(x) : j \in \mathbb{Z}, b \in B, k \in \Gamma, 1 \leq l \leq L \right\}. \quad (2.7)
\end{aligned}$$

Then by (2.7), the theorem is proved. \square

2.2 Finite Groups and Lattices in MSF Composite Dilation Wavelets

It is often beneficial to choose a lattice related to the composite dilation group, B , or the expanding matrix, a . The goal of this section is to show that every full rank lattice is B -admissible when B is a finite group with a fundamental region that is bounded by hyperplanes through 0. This provides a considerable freedom to the composite dilation wavelet system we choose. So we prove the following theorem:

Theorem 2.2. *If B is a finite group whose fundamental region is bounded by hyperplanes through 0, then every lattice $\Gamma = c\mathbb{Z}^n$ is B -admissible.*

First we need a series of lemmas.

Lemma 2.3. *Let G be any group, R be any set such that $R = A \cup B$, $A \cap B = \emptyset$ and R is a G -tiling set for a set V . If $R' = A \cup Bg_o$ for some $g_o \in G$, then R' is a G -tiling set for V .*

Proof.

$$\begin{aligned} \bigcup_{g \in G} R'g &= \bigcup_{g \in G} (A \cup Bg_o)g = \left(\bigcup_{g \in G} Ag \right) \cup \left(\bigcup_{g \in G} Bg_o g \right) \\ &= \left(\bigcup_{g \in G} Ag \right) \cup \left(\bigcup_{g' \in G} Bg' \right) = \bigcup_{g \in G} (A \cup B)g = \bigcup_{g \in G} Rg = V. \end{aligned}$$

Since R is a G -tiling set for V , $R \cap Rg = \emptyset$ for all $g \in G$. Hence $R \cap R(g_o g g_o^{-1}) = \emptyset$.

Then

$$Rg_o \cap Rg_o g = (R \cap R(g_o g g_o^{-1}))g_o = \emptyset g_o = \emptyset.$$

Since $A, B \subset R$, then for all $g \in G$,

$$A \cap Ag \subset R \cap Rg = \emptyset$$

and

$$Bg_o \cap Bg_o g \subset Rg_o \cap Rg_o g = \emptyset.$$

Then for all $g \in G$

$$\begin{aligned} R' \cap R'_g &= (A \cup Bg_o) \cap (A \cup Bg_o)g \\ &= (A \cap Ag) \cup (Bg_o \cap Bg_o g) \\ &= \emptyset. \end{aligned}$$

Therefore, R' is a G -tiling set for V . \square

Notice also that if $R = A \cup B$ is a G -tiling set for V , then $A \cap Bg = \emptyset$ for all $g \in G$. $A, B \subset R$ then $A \cap Bg \subset R \cap Rg = \emptyset$.

Lemma 2.4. *Let F be a region bounded by hyperplanes through 0. Let v be a vector contained in the interior of F . Let K be any compact set. Then K can be translated into F by adding v to K finitely many times.*

Proof. For all $x \in K$, define $L_x = \{x + tv \mid t \in \mathbb{R}\}$. Since v is in the interior of F and no two hyperplanes bounding F are parallel, then $L_x \cap F \neq \emptyset$ for all $x \in K$. Also, for all $x \in K$ define $t_x = \inf\{|x - f| : f \in L_x \cap F\}$. Then for all $t \geq t_x$, $x + \frac{t}{|v|}v \in F$ since $|x + \frac{t}{|v|}v - x| = t \geq t_x$ and F contains its boundary hyperplanes. Since K is compact, $s = \sup_{x \in K} t_x < \infty$. Then let $l = \left\lceil \frac{s}{|v|} \right\rceil$. Then $l < \infty$. For all $x \in K$, $x + lv \in F$ since $l \geq \frac{s}{|v|} \geq \frac{t_x}{|v|}$. Therefore $K + lv \subset F$. Hence we can translate K into F by adding v to K l times where $l < \infty$. \square

The preceding lemma may appear to be obvious. The proof verifies our intuition but also provides us a method of calculating l and shows that l is the minimum integer required to move K into F . This idea is needed more than once in the proof of Theorem 2.2 and could be useful in any possible implementation.

Lemma 2.5. *Suppose F is a fundamental region for a group $B \subseteq GL_n(\mathbb{R})$ and R is a set such that $R \subset F$ and $B_\epsilon(0) \cap F \subset R$ for some $\epsilon > 0$. Let $S = \bigcup_{b \in B} Rb$. Then*

(i) *There exists $\alpha > 0$ such that $B_\alpha(0) \subset S$*

(ii) *if R is starlike with respect to 0, then S is starlike with respect to 0.*

Proof. Since F is a fundamental region for B , then $\mathbb{R}^n = \bigcup_{b \in B} Fb$. For each $b \in B$, define $T_b: F \rightarrow Fb$ by $T_b(x) = xb$. Since $b \in GL_n(\mathbb{R})$, then T_b is an isomorphism.

(i) Since T_b is an isomorphism for each $b \in B$, $T_b(B_\epsilon(0) \cap F)$ is open in Fb . Also, since $0 \in B_\epsilon(0) \cap F$ and $T_b(0) = 0$, then $0 \in T_b(B_\epsilon(0) \cap F)$. Take $\epsilon_b > 0$ such that $(B_{\epsilon_b}(0) \cap Fb) \subset T_b(B_\epsilon(0) \cap F)$. Define $\alpha = \min\{\epsilon_b \mid b \in B\}$. If B is finite, $\alpha > 0$. For all $b \in B$, $B_\alpha(0) \cap Fb \subset T_b(B_\epsilon(0) \cap F)$. Then

$$\bigcup_{b \in B} [B_\alpha(0) \cap Fb] \subset \bigcup_{b \in B} T_b(B_\epsilon(0) \cap F)$$

and, therefore

$$B_\alpha(0) \cap \bigcup_{b \in B} Fb \subset \bigcup_{b \in B} (B_\epsilon(0) \cap F)b.$$

Since $\bigcup_{b \in B} Fb = \mathbb{R}^n$ and $B_\epsilon(0) \cap F \subset R$, then $B_\alpha(0) \subset \bigcup_{b \in B} Rb = S$.

(ii) Suppose R is starlike with respect to 0. Since $0 \in R$, then $0 \in S$. Let $y \in S$. Then there exist $x \in R$ and $b_o \in B$ such that $y = xb_o = T_{b_o}(x)$. Then for all $t \in [0, 1]$, $ty = tT_{b_o}(x) = T_{b_o}(tx)$, since T_{b_o} is an isomorphism. Since R is starlike with respect to 0, then $tx \in R$. Thus $ty = T_{b_o}(tx) \in T_{b_o}(R) \subset \bigcup_{b \in B} Rb$. So, for all $y \in S$, $ty \in S$ for all $t \in [0, 1]$. Therefore, S is starlike with respect to 0. \square

Lemma 2.6. *Suppose F is a region bounded by hyperplanes through 0. Then $F \setminus (F + v)$ is starlike with respect to 0 for any vector v in F .*

Proof. By the definition of F , F is starlike with respect to 0. Also, we observe that

F^C is starlike with respect to 0. Let $x \in F \setminus (F + v)$. Since $x \in F$ and F is starlike with respect to 0, then $tx \in F$ for all $t \in [0, 1]$. Since $x \notin (F + v)$, then $x - v \in F^C$. Since F^C is starlike with respect to 0 then for all $t \in [0, 1]$, $t(x - v) = tx - tv \in F^C$. So for any t , $tx \in F$ and $tx - tv \in F^C$, then $tx \in F \setminus (F + tv) \subset F \setminus (F + v)$. So for all $t \in [0, 1]$, $tx \in F \setminus (F + v)$. Therefore, $F \setminus (F + v)$ is starlike with respect to 0. \square

Now, armed with the preceding lemmas, we can establish Theorem 2.2.

Proof of Theorem 2.2. Let $\Gamma = c\mathbb{Z}^n$ and $\{\gamma_i\}_{i=1}^n = \{\hat{e}_i c^{-1}\}_{i=1}^n$ be a basis for Γ^* . Let $P = \{\sum_{i=1}^n t_i \gamma_i : t_i \in [-\frac{1}{2}, \frac{1}{2}]; i = 1, \dots, n\}$. Let F be a fundamental region for B , then F is bounded by hyperplanes through 0.

Suppose γ_n is in the interior of F . Since P is compact and γ_n is in the interior of F , Lemma 2.4 tells us that there exists a smallest integer m such that $P + m\gamma_n \subset F$. Since P is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$, for any $0 \leq k < l \leq m$ we have $(P + k\gamma_n) \cap (P + l\gamma_n) = \emptyset$. Now Define $A_0 = P \cap F$ and for $j = 1, \dots, m$, iteratively define

$$A_j = [\{(P \setminus \bigcup_{k=0}^{j-1} A_k) + j\gamma_n\} \cap F] - j\gamma_n. \quad (2.8)$$

Then $A_{j_1} \cap A_{j_2} = \emptyset$ for all $j_1 \neq j_2$ and $P = \bigcup_{j=0}^m A_j$.

Let us pause to look at an example of the sets defined by (2.8) in $\hat{\mathbb{R}}^2$. For $B = \langle r_1, r_2 \rangle$, the group generated by reflections through $y = x$ and $y = 0$, and $\Gamma = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} \mathbb{Z}^2$, figure 2.3 shows how we divide P into the sets A_j . A_0 is the set $P \cap F$. After removing A_0 from P , we translate the remaining set by γ_2 and take the intersection with F . When we translate this intersection back into P , we have A_1 . Likewise, we obtain A_2 by (2.8).

Returning to the proof, we define $R = \bigcup_{j=0}^m (A_j + j\gamma_n)$. Then, by lemma 2.2, R is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$. Since $B_\epsilon(0) \subset P$ and $A_0 = P \cap F$, then $B_\epsilon(0) \cap F \subset A_0 \subset R$.

Now we must show the second condition in the definition of B-admissibility is also

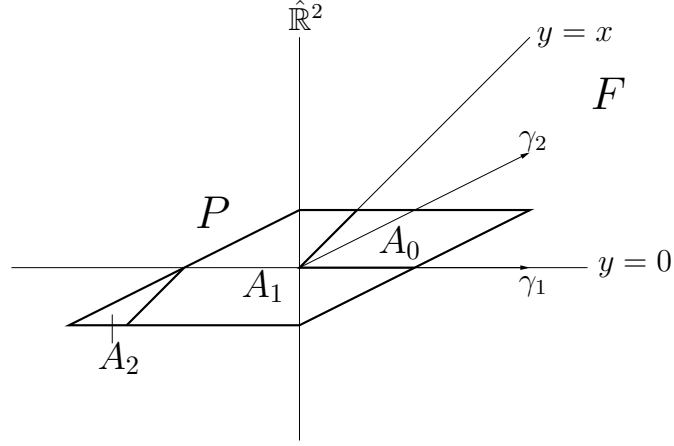


Figure 2.3: $P = \bigcup_{j=0}^m A_j = A_0 \cup A_1 \cup A_2$.

satisfied. First we observe that $\bigcup_{j=0}^m (P + j\gamma_n)$ is starlike with respect to zero.

$$\begin{aligned}
\bigcup_{j=0}^m (P + j\gamma_n) &= \{p + s\gamma_n : p \in P, s \in \{0, \dots, m\}\} \\
&= \left\{ \left(\sum_{i=1}^n t_i \gamma_i \right) + s\gamma_n : t_i \in \left[-\frac{1}{2}, \frac{1}{2}\right]; s \in \{0, \dots, m\} \right\} \\
&= \left\{ \sum_{i=1}^{n-1} t_i \gamma_i + t_n \gamma_n : t_i \in \left[-\frac{1}{2}, \frac{1}{2}\right] \ i = 1, \dots, n-1; t_n \in [0, m] \right\}
\end{aligned}$$

This is clearly a convex set containing 0, hence $\bigcup_{j=0}^m (P + j\gamma_n)$ is starlike with respect to 0. By lemma 2.6, $F \setminus (F + \gamma_n)$ is starlike with respect to 0. Now the intersection of two sets that are starlike with respect to 0 is also starlike with respect to zero. So if the following claim is true, then R is starlike with respect to 0.

Claim 1.

$$R = \left[\bigcup_{j=0}^m (P + j\gamma_n) \right] \cap \left[F \setminus (F + \gamma_n) \right]$$

Proof of Claim: By lemma 2.4, with $\gamma_n \subset F$, there exists a unique $m \in \mathbb{N}$ such that $P + m\gamma_n \subset F$ and $P + (m-1)\gamma_n \cap F^C \neq \emptyset$. Since $A_j \subset P$, then $A_j + j\gamma_n \subset P + j\gamma_n$

for all $j = 1, \dots, m$. Thus $\bigcup_{j=0}^m (A_j + j\gamma_n) \subset \bigcup_{j=0}^m (P + j\gamma_n)$.

Fix any $j \in \{0, \dots, m\}$. Suppose

$$x \in (A_j + j\gamma_n) = \left\{ \left(P \setminus \bigcup_{k=0}^{j-1} A_k \right) + j\gamma_n \right\} \cap F.$$

Then $x \in F$ and $x \in (P \setminus \bigcup_{k=0}^{j-1} A_k) + j\gamma_n$. Therefore $x - j\gamma_n \in P \setminus \bigcup_{k=0}^{j-1} A_k$. Hence $x - j\gamma_n \notin A_{j-1}$. Then $(x - j\gamma_n) + (j-1)\gamma_n \in F^C$ so $x - \gamma_n \in F^C$. Therefore $x \notin (F + \gamma_n)$. So $x \in F$ and $x \notin (F + \gamma_n)$, thus $x \in F \setminus (F + \gamma_n)$. Thus, we have $(A_j + j\gamma_n) \subset F \setminus (F + \gamma_n)$. Since $j \in \{0, \dots, m\}$ was arbitrary, $\bigcup_{j=0}^m (A_j + j\gamma_n) \subset F \setminus (F + \gamma_n)$. Therefore

$$R = \bigcup_{j=0}^m (A_j + j\gamma_n) \subset \left[\bigcup_{j=0}^m (P + j\gamma_n) \right] \cap [F \setminus (F + \gamma_n)].$$

Now, suppose $x \in (\bigcup_{j=0}^m (P + j\gamma_n)) \cap (F \setminus (F + \gamma_n))$. Then $x \in F \setminus (F + \gamma_n)$ so $x \in F$ and $x - \gamma_n \in F^C$. Also, $x \in \bigcup_{j=0}^m (P + j\gamma_n)$. Since P is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$, then $P + j\gamma_n \cap P + k\gamma_n = \emptyset$ for any $j \neq k$. So there exists a unique $j' \in \{0, \dots, m\}$ such that $x \in P + j'\gamma_n$. So, $x \in (P + j'\gamma_n) \cap F$.

Since $x - \gamma_n \in F^C$ then $x - l\gamma_n \in F^C$ for all $l \in \{1, \dots, j'\}$. Since $x \in P + j'\gamma_n$ then $x - l\gamma_n \in P + (j' - l)\gamma_n$ for all $l \in \{1, \dots, j'\}$. Hence $(x - l\gamma_n) \in [P + (j' - l)\gamma_n] \cap F^C$. Thus for all $l = 1, \dots, j'$ we have $x - l\gamma_n \notin [P + (j' - l)\gamma_n] \cap F$. Now

$$\begin{aligned} A_{j'-l} + (j' - l)\gamma_n &= \left[\left(P \setminus \bigcup_{k=0}^{j'-l-1} A_k \right) + (j' - l)\gamma_n \right] \cap F \\ &\subset (P + (j' - l)\gamma_n) \cap F. \end{aligned}$$

Hence, for all $l = 1, \dots, j'$ we have $x - l\gamma_n \notin A_{j'-l} + (j' - l)\gamma_n$. Then $x \notin A_{j'-l} + j'\gamma_n$.

Thus $x \notin \bigcup_{k=0}^{j'-1} (A_k + j'\gamma_n)$. So we have

$$x \in (P + j'\gamma_n) \setminus \bigcup_{k=0}^{j'-1} (A_k + j'\gamma_n) = (P \setminus \bigcup_{k=0}^{j'-1} (A_k)) + j'\gamma_n.$$

We already have $x \in F$, thus

$$x \in [(P \setminus \bigcup_{k=0}^{j'-1} (A_k)) + j'\gamma_n] \cap F = A_{j'} + j'\gamma_n.$$

So for all $x \in (\bigcup_{j=0}^m (P + j\gamma_n)) \cap (F \setminus (F + \gamma_n))$ there exists $j' \in \{0, \dots, m\}$ such that $x \in A_{j'} + j'\gamma_n$. Therefore

$$\left[\bigcup_{j=0}^m (P + j\gamma_n) \right] \cap [F \setminus (F + \gamma_n)] \subset \bigcup_{j=0}^m (A_j + j\gamma_n) = R.$$

Therefore claim 1 is true.

Let us examine claim 1 with our two dimensional example. For $B = \langle r_1, r_2 \rangle$, the group generated by reflections through $y = x$ and $y = 0$, and $\Gamma = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} \mathbb{Z}^2$, figure 2.4 shows how we build R as a union of translations of the sets A_j . In this figure, we can also see how R is the intersection of the set $F \setminus (F + \gamma_n)$, with a union of translates of the parallelogram, P . In figure 2.4, $R = \bigcup_{j=0}^3 (A_j + j\gamma_2) = \left[\bigcup_{j=0}^3 (P + j\gamma_2) \right] \cap [F \setminus (F + \gamma_2)]$.

Having established claim 1, we have determined that R is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$, R is starlike with respect to 0, $R \subset F$, and $B_\epsilon(0) \cap F \subset R$. Define $S = \bigcup_{b \in B} Rb$. Then by lemma 2.5, S is a starlike neighborhood of zero. Therefore Γ is B -admissible.

Our two dimensional example provides insight into how we constructed the sets R and S . Figure 2.5 depicts the sets R and S with our running example taking $B = \langle r_1, r_2 \rangle$ and $\Gamma = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} \mathbb{Z}^2$.

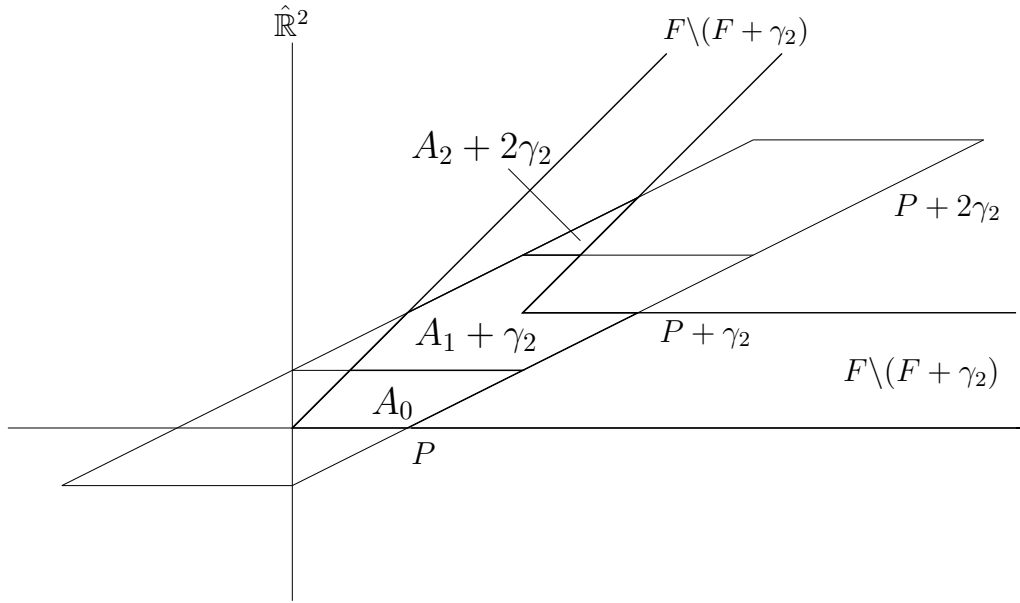


Figure 2.4: $R = A_0 \cup (A_1 + \gamma_2) \cup (A_2 + 2\gamma_2) = \left[\bigcup_{j=0}^2 (P + j\gamma_2) \right] \cap [F \setminus (F + \gamma_2)]$.

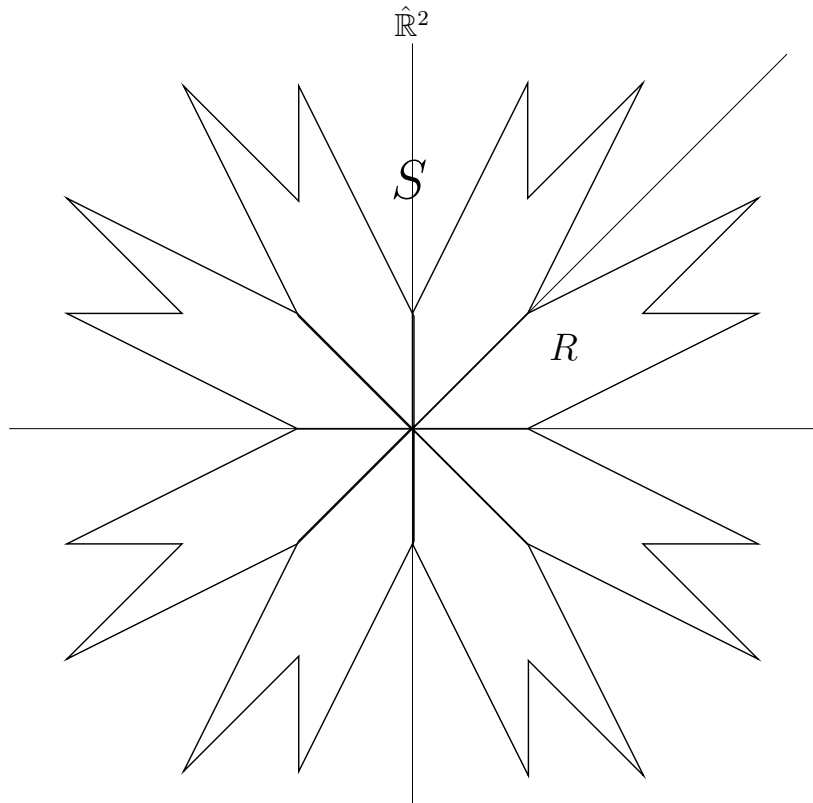


Figure 2.5: $S = \bigcup_{b \in B} Rb$.

We assumed that γ_n was in the interior of F . If this is not the case, there are four other possibilities:

1. There exists $b \in B$ such that γ_n is in the interior of Fb .
2. γ_n is not in the interior of Fb for all $b \in B$, but there exists $k \in \{1, \dots, n-1\}$ such that γ_k is in the interior of Fb for some $b \in B$.
3. For all $k \in \{1, \dots, n\}$, γ_k is in the boundary of Fb_0 for a fixed $b_0 \in B$.
4. For all $k \in \{1, \dots, n\}$, γ_k is in the boundary of Fb for some $b \in B$.

In case 1, simply take Fb as the fundamental region and proceed as above. In case 2, take γ_k and $b \in B$ such that γ_k is in the interior of Fb . Then, relabel the basis and proceed as above. For the third case, the parallelepiped spanned by the basis is R and choose Fb_0 as the fundamental region. The little that remains of the proof remains the same.

The final case requires very little as well. First choose the fundamental region, F , such that γ_n is in the boundary of F . Choose a new basis $\{\tilde{\gamma}_i\}_{i=1}^n$ with $\tilde{\gamma}_i = \pm\gamma_i$ for all $i = 1, \dots, n$ such that each $\tilde{\gamma}_i$ and the fundamental region, F , lie in the same half space determined by the hyperplane containing γ_n . Now define $P = \{\sum_{i=1}^n t_i \tilde{\gamma}_i : 0 \leq t_i \leq 1\}$. We may now follow the same procedure in the proof since there exists a minimal $m \in \mathbb{Z}$ such that $P + m\tilde{\gamma}_n \subset F$. Therefore, theorem 2.2 is proved. \square

The set of all finite Coxeter groups is a subset of the family of groups that have a fundamental region bounded by hyperplanes through 0. This is presented in Chapter 4 of Finite Reflection Groups by Grove and Benson [7]. This provides the following:

Corollary 2.7. *If B is a finite Coxeter Group, then every lattice $\Gamma = c\mathbb{Z}^n$ is B -admissible.*

Corollary 2.8. *If B is a finite Rotation Group, then every lattice $\Gamma = c\mathbb{Z}^n$ is B -*

admissible.

2.3 Existence of MSF Composite Dilation

Wavelets in $L^2(\mathbb{R}^n)$.

This section shows that for any dimension, we can always find a composite dilation wavelet system generating an orthonormal basis for $L^2(\mathbb{R}^n)$. In fact, we can do so with the same freedom as the previous section. We begin the section by showing that $2(I_n)$ is always (B, Γ) -admissible when we chose a group, B , and a full rank lattice, Γ , according to the preceding section.

Theorem 2.9. *If B is a group whose fundamental region is bounded by n distinct hyperplanes through the origin and Γ is any full rank lattice ($\Gamma = c\mathbb{Z}^n$), then $a = 2I_n$ is (B, Γ) -admissible.*

Proof. Recall from Theorem 2.2 that $P = \{\sum_{i=1}^n t_i \gamma_i : t_i \in [-\frac{1}{2}, \frac{1}{2}], i = 1, \dots, n\}$ where $\{\gamma_i\}_{i=1}^n = \{\hat{e}_i c^{-1}\}_{i=1}^n$ are the basis vectors for a Γ^* . We assume that we have already constructed the set R according to theorem 2.2 so that $\{\gamma_i\}_{i=1}^n$ is ordered such that γ_n is in the interior of F , a fundamental region of $\hat{\mathbb{R}}^n$ for B . P is simply the parallelepiped formed by the basis vectors $\{\gamma_i\}_{i=1}^n$ and then translated by $-\frac{1}{2} \sum_{i=1}^n \gamma_i$. Therefore, P is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$.

We make the following definitions to be used throughout the proof:

$$\Omega = \bigcup_{j \in \mathbb{Z}} (P + j\gamma_n) = \left\{ \sum_{i=1}^{n-1} t_i \gamma_i + \beta_n \gamma_n : t_i \in [-\frac{1}{2}, \frac{1}{2}], \beta_n \in \mathbb{R} \right\} \quad (2.9)$$

$$K_i = \left\{ 0\gamma_i + \sum_{\substack{j=1 \\ j \neq i}}^n \beta_j \gamma_j : \beta_j \in \mathbb{R} \forall j \right\} \text{ for } 1 \leq i \leq n-1. \quad (2.10)$$

Then $\{K_i\}_{i=1}^{n-1}$ is a set of hyperplanes, each formed by the span of the basis vectors

not including the i th vector. Clearly $L := \bigcap_{i=1}^{n-1} K_i = \{\beta_n \gamma_n : \beta_n \in \mathbb{R}\}$ is the line defined by γ_n . We observe that Ω contains L as its center (with respect to Γ^*). We see immediately from the definition of Ω that the hyperplanes $\{K_i \pm \frac{1}{2}\gamma_i\}_{i=1}^{n-1}$ bound Ω in the sense that Ω is the nonempty intersection of the half-planes determined by each hyperplane $K_i \pm \frac{1}{2}\gamma_i$. (In the remaining discussion we say a set L is bounded by a collection of hyperplanes $\{H_1, \dots, H_s\}$ if and only if L is the nonempty intersection of the half-spaces defined by the collection $\{H_1, \dots, H_s\}$.)

Now we proceed by dividing the region Ω into 2^{n-1} disjoint chambers by cutting Ω in half with each of the hyperplanes K_i . For each $i = 1, \dots, n-1$, K_i divides Ω into two regions

$$\left\{ \pm s_i \gamma_i + \sum_{\substack{j=1 \\ j \neq i}}^{n-1} t_j \gamma_j + \beta_n \gamma_n : s_i \in [0, \frac{1}{2}], t_j \in [-\frac{1}{2}, \frac{1}{2}], \beta_n \in \mathbb{R} \right\}.$$

We can further divide each of these regions into two more regions by inserting one of the other hyperplanes from $\{K_i\}_{i=1}^{n-1}$. Continuing in this fashion, we divide Ω into disjoint chambers.

Let V denote the set of all $(n-1) \times 1$ vectors with entries from $\{-1, 1\}$. Then $|V| = 2^{n-1}$. For each $v \in V$, define

$$\Omega_v = \left\{ \sum_{i=1}^{n-1} v_i s_i \gamma_i + \beta_n \gamma_n : v = (v_1, \dots, v_{n-1}), s_i \in [0, \frac{1}{2}], \beta_n \in \mathbb{R} \right\} \quad (2.11)$$

Then $\Omega = \bigcup_{v \in V} \Omega_v$. So we have partitioned Ω into 2^{n-1} disjoint chambers of equal measure. By inspection we see that for each $v \in V$, Ω_v is the region bounded by the hyperplanes $\{K_i\}_{i=1}^{n-1}$ and $\{K_i + v_i \frac{1}{2}\gamma_i\}_{i=1}^{n-1}$.

Here we pause to examine the regions P , Ω , and Ω_v along with the hyperplanes K_i and $K_i \pm \frac{1}{2}\gamma_i$ in the two dimensional example that we used in the proof of theorem

2.2. For $\Gamma = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} \mathbb{Z}^2$, we show in figure 2.6 the sets P , Ω , Ω_{-1} , and Ω_1 . These sets are defined in part by the hyperplanes K_1 and $K_1 \pm \frac{1}{2}\gamma_1$.

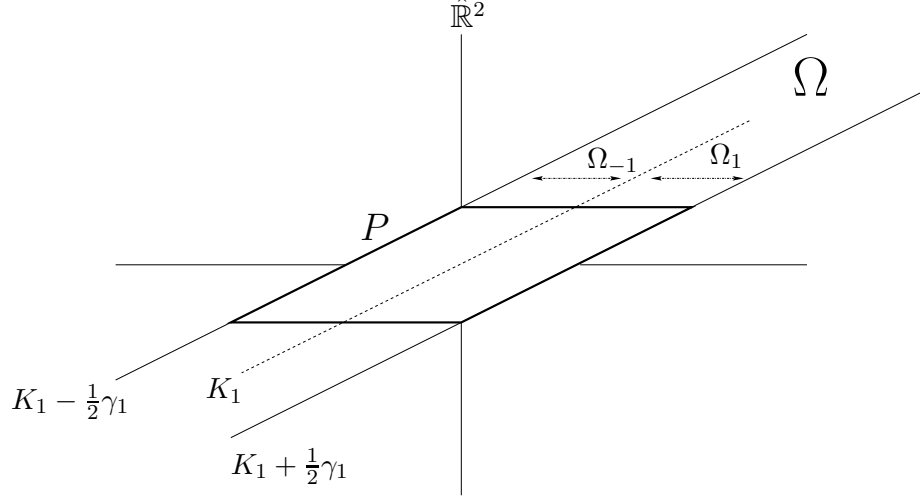


Figure 2.6: $\Omega = \bigcup_{j \in \mathbb{Z}} (P + j\gamma_2)$ and $\Omega = \Omega_{-1} \cup \Omega_1$.

Returning to our proof, we recall from theorem 2.2 that for some $m \in \mathbb{N}$

$$R = \left[\bigcup_{j=0}^m (P + j\gamma_n) \right] \cap [F \setminus (F + \gamma_n)]$$

so

$$R \subset \bigcup_{j \in \mathbb{Z}} (P + j\gamma_n) = \Omega.$$

Moreover, we see that

$$\begin{aligned} \bigcup_{j \in \mathbb{Z}} (R + j\gamma_n) &= \bigcup_{j \in \mathbb{Z}} \left(\left\{ \left[\bigcup_{l=0}^m (P + l\gamma_n) \right] \cap [F \setminus (F + \gamma_n)] \right\} + j\gamma_n \right) \\ &= \left\{ \bigcup_{j \in \mathbb{Z}} \left(\left[\bigcup_{l=0}^m (P + l\gamma_n) \right] + j\gamma_n \right) \right\} \cap \left\{ \bigcup_{j \in \mathbb{Z}} ([F \setminus (F + \gamma_n)] + j\gamma_n) \right\}. \end{aligned} \tag{2.12}$$

Now the first term in (2.12) is simply Ω .

$$\begin{aligned}
\bigcup_{j \in \mathbb{Z}} \left(\left[\bigcup_{l=0}^m (P + l\gamma_n) \right] + j\gamma_n \right) &= \bigcup_{j \in \mathbb{Z}} \bigcup_{l=1}^m (P + (j+l)\gamma_n) \\
&= \bigcup_{j' \in \mathbb{Z}} (P + j'\gamma_n) \\
&= \Omega.
\end{aligned} \tag{2.13}$$

Also, the second term in (2.12) is $\hat{\mathbb{R}}^n$.

$$\begin{aligned}
\bigcup_{j \in \mathbb{Z}} ([F \setminus (F + \gamma_n)] + j\gamma_n) &= \bigcup_{j \in \mathbb{Z}} (\{F + j\gamma_n\} \setminus \{F + (j+1)\gamma_n\}) \\
&= \bigcup_{j \in \mathbb{Z}} (F + j\gamma_n)
\end{aligned} \tag{2.14}$$

since each portion that is dropped is then recovered in the union. Now let $x \in \hat{\mathbb{R}}^n$. Then $\{x\}$ is a compact set. From lemma 2.4 we have $l \in \mathbb{N}$ such that $x + l\gamma_n \in F$. Then $x \in F - l\gamma_n$. Therefore

$$\bigcup_{j \in \mathbb{Z}} (F + j\gamma_n) = \hat{\mathbb{R}}^n.$$

Hence, from (2.13) and (2.14) we have

$$\bigcup_{j \in \mathbb{Z}} (R + j\gamma_n) = \Omega \cap \hat{\mathbb{R}}^n = \Omega.$$

F is a fundamental region for the group B and therefore is bounded by hyperplanes through the origin. We let H denote the union of the portion of these hyperplanes that form the boundary of the fundamental region, F . Since $R \subset F \setminus (F + \gamma_n)$, then R lies in the region of Ω bounded by H and $H + \gamma_n$, i.e. $R \subset \Omega \cap \{F \setminus (F + \gamma_n)\}$. Now we can describe R as the region bounded by $\{K_i + v_i \frac{1}{2} \gamma_i\}_{i=1}^{n-1}$, H , and $H + \gamma_n$. For all

$v \in V$ we define

$$R_v = R \cap \Omega_v. \quad (2.15)$$

Since $\Omega = \bigcup_{v \in V} \Omega_v$ then

$$R = R \cap \left(\bigcup_{v \in V} \Omega_v \right) = \bigcup_{v \in V} (R \cap \Omega_v) = \bigcup_{v \in V} R_v \quad (2.16)$$

Here we see that we have a partition of R into 2^{n-1} disjoint chambers $\{R_v\}_{v \in V}$. Since Ω_v is bounded by the hyperplanes $\{K_i\}_{i=1}^{n-1}$ and $\{K_i + v_i \frac{1}{2} \gamma_i\}_{i=1}^{n-1}$, then R_v is bounded by the hyperplanes $\{K_i\}_{i=1}^{n-1}$, $\{K_i + v_i \frac{1}{2} \gamma_i\}_{i=1}^{n-1}$, H , and $H + \gamma_n$.

Continuing with our two-dimensional example where $\Gamma = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} \mathbb{Z}^2$ and $B = \langle r_1, r_2 \rangle$, figure 2.7 shows the regions R and R_v (where $v \in \{1, -1\}$) along with the hyperplanes K_1 and $K_1 \pm \frac{1}{2} \gamma_1$.

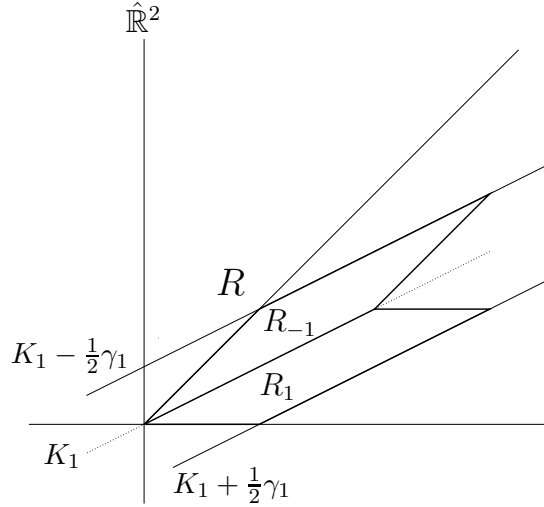


Figure 2.7: $R = R_{-1} \cup R_1$.

Now we will describe how the expanding matrix $a = 2I_n$ acts on R by observing that we may look at its effect on the hyperplanes bounding R . We will also determine how $2I_n$ acts on the hyperplanes bounding each R_v . Since R is bounded by the hyperplanes $\{K_i \pm \frac{1}{2} \gamma_i\}_{i=1}^{n-1}$, then $R(2I_n)$ is bounded by the hyperplanes $\{K_i \pm 2 \frac{1}{2} \gamma_i\}_{i=1}^{n-1}$

or $\{K_i \pm \gamma_i\}_{i=1}^{n-1}$ since every hyperplane is invariant under $2I_n$ and the translation of the hyperplane is simply multiplied by 2:

$$K_i(2I_n) = \left\{ 0\gamma_i + \sum_{\substack{j=1 \\ j \neq i}}^n \beta_j \gamma_j \right\} (2I_n) = \left\{ 2(0\gamma_i) + \sum_{\substack{j=1 \\ j \neq i}}^n 2\beta_j \gamma_j : \beta_j \in \mathbb{R} \right\} = K_i$$

$$\left(K_i \pm \frac{1}{2}\gamma_i \right) (2I_n) = K_i(2I_n) \pm 2\frac{1}{2}\gamma_i = K_i \pm \gamma_i$$

Each hyperplane forming the boundary of F is invariant under $2I_n$. Thus the union of the hyperplanes forming the boundary of F is also invariant under $2I_n$. Therefore, the boundary of F , which we call H , is invariant under $2I_n$. Hence $H(2I_n) = H$ and $(H + \gamma_n)(2I_n) = H + 2\gamma_n$. Now R is also bounded by H and $H + \gamma_n$, thus $R(2I_n)$ is bounded by H and $H + 2\gamma_n$. So we see that $R(2I_n)$ is bounded by the hyperplanes $\{K_i \pm \gamma_i\}_{i=1}^{n-1}$, H , and $H + 2\gamma_n$.

Similarly, we examine the bounding hyperplanes of R_v for all $v \in V$. For each $v = (v_1, \dots, v_n) \in V$ the bounding hyperplanes are simply the subset defined by v of those hyperplanes bounding R and the addition of the hyperplanes $\{K_i\}_{i=1}^{n-1}$. Of course, R_v is also bounded by H and $H + \gamma_n$. Therefore, the bounding hyperplanes of $R_v(2I_n)$ are $\{K_i\}_{i=1}^{n-1}$, $\{K_i + v_i \gamma_i\}_{i=1}^{n-1}$, H , and $H + 2\gamma_n$. We see that this is also a parallelepiped. So from equation (2.16) we see that R is a union of parallelepipeds.

Returning to R_v , we can further divide the parallelepiped $R_v(2I_n)$ by reinserting the original hyperplanes bounding R_v , namely $\{K_i + v_i \frac{1}{2}\gamma_i\}_{i=1}^{n-1}$. These new regions are simply translations of R_v . Thus, $R_v(2I_n)$ is a union of translates of the parallelepiped R_v .

Let W be the set of all $n \times 1$ vectors with entries in $\{0, 1\}$,

$$W = \{w = (w_1, \dots, w_n) : w_i \in \{0, 1\}\}.$$

Then $|W| = 2^n$. These translates of R_v that form $R_v(2I_n)$ will be defined as R_{vw} . The translation is not necessarily in any direction defined by the lattice basis. However, they are related to this basis in that we must translate in directions along the bounding hyperplanes H to the intersections with $\{K_i + v_i \frac{1}{2} \gamma_i\}_{i=1}^{n-1}$. The following describes these translations.

Fix $v \in V$. Let H_v be the portion of H that is bounding R_v . That is $H_v = H \cap \Omega_v$. Now define $h_{v(i)} \in H$ as the unique vector originating at the origin that will translate H_v to the portion of H in the strip bounded by $K_i + v_i \frac{1}{2} \gamma_i$ and $K_i + v_i \gamma_i$. Since K_i is only defined for $i = 1, \dots, n-1$, $h_{v(n)}$ is not defined. For clarity, we can define $h_{v(n)} = \gamma_n$. That is $h_{v(i)}$ is the unique vector such that

$$H_v + h_{v(i)} = H \cap \left(\Omega_v + v_i \frac{1}{2} \gamma_i \right).$$

Now we define the regions R_{vw} :

$$R_{vw} = R_v + \sum_{i=1}^{n-1} w_i h_{v(i)} + w_n \gamma_n. \quad (2.17)$$

We must examine the bounding hyperplanes of R_{vw} for a fixed $v \in V$ and an arbitrary $w \in W$. We know that R_v is bounded by the hyperplanes $\{K_i\}_{i=1}^{n-1}$, $\{K_i + v_i \frac{1}{2} \gamma_i\}_{i=1}^{n-1}$, H , and $H + \gamma_n$. So for a fixed $v \in V$ and any $w \in W$, R_{vw} is bounded by the hyperplanes

$$\{K_i + w_i h_{v(i)} + w_n \gamma_n\}_{i=1}^{n-1} \quad (2.18)$$

$$\left\{ K_i + v_i \frac{1}{2} \gamma_i + w_i h_{v(i)} + w_n \gamma_n \right\}_{i=1}^{n-1} \quad (2.19)$$

$$H + \sum_{i=1}^{n-1} w_i h_{v(i)} + w_n \gamma_n \quad (2.20)$$

$$H + \gamma_n + \sum_{i=1}^{n-1} w_i h_{v(i)} + w_n \gamma_n \quad (2.21)$$

We know $h_{v(i)}$ points to the subspace $\{K_i + v_i \frac{1}{2} \gamma_i\} \cap H$, thus a translation by $w_i h_{v(i)}$ applied to the region bounded by $\{K_i\}$ and $\{K_i + v_i \frac{1}{2} \gamma_i\}$ is a translation by $w_i v_i \frac{1}{2} \gamma_i$. By the definition of K_i (2.10), $K_i + w_n \gamma_n = K_i$ for all $i = 1, \dots, n-1$. Therefore (2.18) and (2.19) become

$$\left\{ K_i + w_i v_i \frac{1}{2} \gamma_i \right\}_{i=1}^{n-1} \quad (2.22)$$

$$\left\{ K_i + v_i (w_i + 1) \frac{1}{2} \gamma_i \right\}_{i=1}^{n-1} \quad (2.23)$$

Since $v \in V$ is fixed, the portion of H in the appropriate subspace determined by v will be invariant under translations by $w_i h_{v(i)}$. Therefore (2.20) and (2.21) become

$$H + w_n \gamma_n \quad (2.24)$$

$$H + (w_n + 1) \gamma_n. \quad (2.25)$$

Now the set $\bigcup_{w \in W} R_{vw}$ will be bounded by the outermost bounding hyperplanes, those formed by taking $w_i = 0$ for all $i = 1, \dots, n$ in equations (2.18) and (2.20), and by taking $w_i = 1$ for all $i = 1, \dots, n$ in equations (2.19) and (2.21). This tells us that $\bigcup_{w \in W} R_{vw}$ is the non-empty intersection of the half-spaces determined by the hyperplanes

$$\{K_i\}_{i=1}^{n-1}, \{K_i + v_i \gamma_i\}_{i=1}^{n-1}, H, H + 2\gamma_n. \quad (2.26)$$

The region bounded these hyperplanes is precisely $R_v(2I_n)$. Therefore we have established

$$R_v(2I_n) = \bigcup_{w \in W} R_{vw} \quad (2.27)$$

$$R(2I_n) = \bigcup_{v \in V} R_v(2I_n) = \bigcup_{v \in V} \bigcup_{w \in W} R_{vw} \quad (2.28)$$

In two dimensions $V = \{-1, 1\}$ and $W = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$. Continuing with our example where $B = \langle r_1, r_2 \rangle$, the group generated by reflections through $y = x$ and $y = 0$, and $\Gamma = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} \mathbb{Z}^2$, figure 2.8 shows us how $R(2I_n) = \bigcup_{v \in V} \bigcup_{w \in W} R_{vw}$. The vectors $h_{-1(1)} = h_{-1}$ and $h_{1(1)} = h_1$ are also shown.

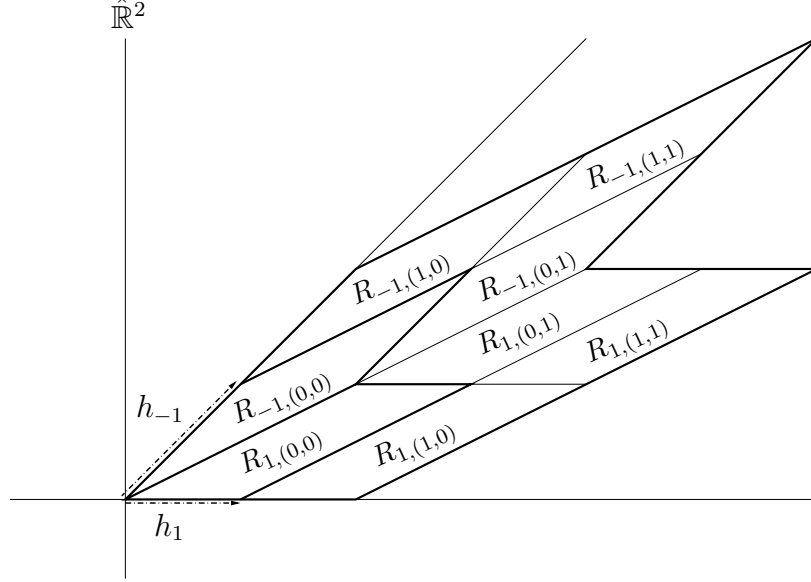


Figure 2.8: $R(2I_n) = \bigcup_{v \in V} \bigcup_{w \in W} R_{vw}$.

Now we have established the necessary understanding and notation of the action of the expanding matrix $a = 2I_n$ on the region R . We complete the proof by showing that each of the sets in the collection $\{\bigcup_{v \in V} R_{vw} : w \in W\}$ satisfies (i) and (ii) in definition 2.4.

Claim 2. For a fixed $w \in W$, $\bigcup_{v \in V} R_{vw}$ is a Γ^* -tiling of $\hat{\mathbb{R}}^n$.

Proof of claim: First we establish that $\bigcup_{v \in V} R_{vw}$ has the necessary measure to be a Γ^* -tiling of $\hat{\mathbb{R}}^n$, i.e. that $m(\bigcup_{v \in V} R_{vw}) = m(P) = \prod_{i=1}^n |\gamma_i|$. For each $v \in V$, R_{vw} is a parallelepiped and therefore we can compute its measure by taking the products of the distances between its bounding hyperplanes in the directions of the basis vectors

$\{\gamma_i\}_{i=1}^n$. Since $|\{H + w_n\gamma_n\} - \{H + (w_n + 1)\gamma_n\}| = |\gamma_n|$, then for each $v \in V$

$$\begin{aligned}
m(R_{vw}) &= |\gamma_n| \prod_{i=1}^{n-1} \left| \left\{ K_i + w_i v_i \frac{1}{2} \gamma_i \right\} - \left\{ K_i + v_i (w_i + 1) \frac{1}{2} \gamma_i \right\} \right| \\
&= |\gamma_n| \prod_{i=1}^{n-1} \left| v_i \frac{1}{2} \gamma_i \right| \\
&= |\gamma_n| \prod_{i=1}^{n-1} \frac{1}{2} |v_i| |\gamma_i| \\
&= \frac{1}{2^{n-1}} \prod_{i=1}^n |\gamma_i| \\
&= \frac{1}{2^{n-1}} m(P)
\end{aligned}$$

Then, since $\bigcup_{v \in V} R_{vw}$ is a disjoint union, we have

$$\begin{aligned}
m\left(\bigcup_{v \in V} R_{vw}\right) &= \sum_{v \in V} m(R_{vw}) = \sum_{v \in V} \frac{1}{2^{n-1}} m(P) \\
&= |V| \frac{1}{2^{n-1}} m(P) = 2^{n-1} \frac{1}{2^{n-1}} m(P) = m(P)
\end{aligned}$$

Therefore, $\bigcup_{v \in V} R_{vw}$ has the appropriate measure to be a Γ^* -tiling of $\hat{\mathbb{R}}^n$.

We now must show that any Γ^* translates of the elements in the union are disjoint. We will establish this by examining the translates in the γ_i directions individually. We will see that no translates in the γ_i direction of any element in the union can intersect any of the other elements in the union.

Claim 3. Let $u, v \in V$ such that $u_i \neq v_i$. For any $l_i \in \mathbb{Z}$ and a fixed $w \in W$, $(R_{uw} + l_i \gamma_i) \cap R_{vw} = \emptyset$.

Proof of claim: Since $u_i \neq v_i$ then $u_i = -v_i$. Without loss of generality, assume $v_i = 1$. Then R_{uw} is contained in the region

$$\left\{ K_i + u_i w_i \frac{1}{2} \gamma_i + t u_i \frac{1}{2} \gamma_i : t \in [0, 1] \right\} = \left\{ K_i - v_i w_i \frac{1}{2} \gamma_i - t v_i \frac{1}{2} \gamma_i : t \in [0, 1] \right\}$$

$$= \left\{ K_i - (w_i + t) \frac{1}{2} \gamma_i : t \in [0, 1] \right\}.$$

So $R_{uw} + l_i \gamma_i$ is contained in the region $\{K_i + [l_i - (w_i + t) \frac{1}{2}] \gamma_i : t \in [0, 1]\}$. Also, R_{vw} is contained in the region $\{K_i + (w_i + t) \frac{1}{2} \gamma_i : t \in [0, 1]\}$. If there exists a point $x \in (R_{uw} + l_i \gamma_i) \cap R_{vw}$ then $l_i - (w_i + t) \frac{1}{2} = (w_i + t) \frac{1}{2}$ so $l_i = w_i + t$. Since $t \in [0, 1], w_i \in \{0, 1\}, l_i \in \mathbb{Z}$, then $l_i \in \{0, 1, 2\}$.

Suppose $l_i = 0$. Then $w_i + t = 0$ so $t = -w_i$. Since $w_i \in \{0, 1\}$ and $t \in [0, 1]$ then $t = w_i = 0$. In this case, $x \in K_i$ is a point in the boundary of each set R_{uw} and R_{vw} .

Suppose $l_i = 1$. Then $1 - t = w_i \in \{0, 1\}$. If $w_i = 0$, then $t = 1$. If $w_i = 1$, then $t = 0$. In either case, $x \in K_i + \frac{1}{2} \gamma_i$, a bounding hyperplane.

Suppose $l_i = 2$. Then $w_i + t = 2$. Since $w_i, t \leq 1$, then $w_i = t = 1$. In this case, $x \in K_i + \gamma_i$ is a point in a hyperplane bounding the sets.

Therefore, we see that if $u, v \in V$ such that $u_i \neq v_i$, then for any $l_i \in \mathbb{Z}$, $(R_{uw} + l_i \gamma_i) \cap R_{vw} = \emptyset$. Therefore, claim 3 is valid.

Claim 4. Suppose $u, v \in V$ such that $u_i = v_i$. Let $l_i \in \mathbb{Z}$ and fix $w \in W$. If $u \neq v$, then $(R_{uw} + l_i \gamma_i) \cap R_{vw} = \emptyset$.

Proof of claim: $R_{uw} + l_i \gamma_i$ is contained in the region $\{K_i + v_i w_i \frac{1}{2} \gamma_i + t v_i \frac{1}{2} \gamma_i + l_i \gamma_i : t \in [0, 1]\}$ and R_{vw} is contained in the region $\{K_i + v_i w_i \frac{1}{2} \gamma_i + t v_i \frac{1}{2} \gamma_i : t \in [0, 1]\}$. Let $x \in (R_{uw} + l_i \gamma_i) \cap R_{vw}$. Then there exists $t_x \in [0, 1]$ such that $l_i + v_i (w_i + t_x) \frac{1}{2} = v_i (w_i + t_x) \frac{1}{2}$. Thus, $l_i = 0$. Therefore, either $u = v$ or $u_j \neq v_j$ for some $j \neq i$ in which case claim 3 verifies that they are essentially disjoint. Thus, claim 4 holds.

From claims 2 - 4 we see that for every $w \in W$, $\bigcup_{v \in V} R_{vw}$ is a Γ^* -tiling for $\hat{\mathbb{R}}^n$.

To establish part (ii) of definition 2.4 we define $S = \bigcup_{b \in B} Rb$ as in the proof of theorem 2.2. Then

$$S(2I_n) = \left(\bigcup_{b \in B} Rb \right) (2I_n)$$

$$\begin{aligned}
&= \bigcup_{b \in B} Rb(2I_n) \\
&= \bigcup_{b \in B} R(2I_n)b
\end{aligned} \tag{2.29}$$

Therefore, using equations (2.28) and (2.29) we get

$$\begin{aligned}
S(2I_n) \setminus S &= \left\{ \bigcup_{b \in B} R(2I_n)b \right\} \setminus \left\{ \bigcup_{b \in B} Rb \right\} \\
&= \bigcup_{b \in B} (R(2I_n) \setminus R) b \\
&= \bigcup_{b \in B} \left(\left\{ \bigcup_{v \in V} \bigcup_{w \in W} R_{vw} \right\} \setminus \left\{ \bigcup_{v \in V} R_{v0} \right\} \right) b \\
&= \bigcup_{b \in B} \left(\bigcup_{v \in V} \bigcup_{w \in W} R_{vw} \setminus R_{v0} \right) b \\
&= \bigcup_{b \in B} \left(\bigcup_{\substack{w \in W \\ w \neq (0, \dots, 0)}} \bigcup_{v \in V} R_{vw} \right) b
\end{aligned} \tag{2.30}$$

Then equation (2.30) tells us that $\bigcup_{\substack{w \in W \\ w \neq (0, \dots, 0)}} \bigcup_{v \in V} R_{vw}$ is a B -tiling set of $S(2I_n) \setminus S$.

Therefore claims 2, 3, 4 and equation (2.30) verify that for each $w \in W$ the regions

$$\left\{ \bigcup_{v \in V} R_{vw} : w \in W \setminus (0, \dots, 0) \right\}$$

satisfy (i) and (ii) from definition 2.4.

Therefore, $a = 2I_n$ is (B, Γ) -admissible.

We began the proof assuming that we had ordered the basis of Γ^* such that γ_n is in the interior of the fundamental region, F . The other possible cases overlap those from the proof of theorem 2.2. If γ_n is not in the interior of the fundamental region, then each element of the basis $\{\gamma_i\}_{i=1}^n$ lies in a hyperplane bounding the fundamental region, F . This actually simplifies the proof as each K_i is a hyperplane bounding the set R . We eliminate the need to divide the region R into sectors using the hyperplanes

$K_i + v_i \frac{1}{2} \gamma_i$. This is a specific case of the preceding proof. \square

Theorems 2.2 and 2.9 were proved by subtracting the vector $\sum_{i=1}^n \frac{1}{2} \gamma_i$ from the parallelepiped formed by the basis of Γ^* . This is not the most general setting in which we could have proved these theorems. Let \tilde{P} be the parallelepiped formed by the basis, $\{\gamma_i\}_{i=1}^n$, of Γ^* . Choose a vector $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $\tilde{P} - \sum_{i=1}^n \alpha_i \gamma_i$ contains an open neighborhood of the origin. Define $P = \tilde{P} - \sum_{i=1}^n \alpha_i \gamma_i$. Let K_i remain the same from the proof of theorem 2.9. Then P and subsequently Ω are bounded by the hyperplanes $\{K_i - \alpha_i \gamma_i\}$ and $\{K_i + (1 - \alpha_i) \gamma_i\}$. Keeping the same definition for the set V of $1 \times (n - 1)$ row vectors with entries from $\{1, -1\}$, then we can define these hyperplanes by

$$\left\{ K_i + \left(\frac{1 + v_i}{2} - \alpha_i \right) \gamma_i \right\}.$$

We may then proceed to define R , Ω_v , R_v , $h_{v(i)}$, etc. This slightly more difficult notation does not prevent us from proving theorems 2.2 and 2.9. In our proofs, we simplified the notation by taking $\alpha_i = \frac{1}{2}$ for all $i = 1, \dots, n$ so that $\left(\frac{1 + v_i}{2} - \alpha_i \right) = v_i \frac{1}{2}$.

This proof of 2.9 highlights three things. First of all, in general, it is not that easy to demonstrate that a matrix is (B, Γ) -admissible when we do not specify B and Γ . On the other hand, if we specify B and Γ , it may not be any easier. Second, the freedom of choosing a lattice does not complicate things too much. In potential applications, one may need to choose the lattice so that R closely resembles the geometry of the signal in the frequency domain. Finally, since we proved that $2I_n$ is always (B, Γ) -admissible, we have the following theorem:

Theorem 2.10. *If B is any group with fundamental region bounded by n hyperplanes through the origin and $\Gamma = c\mathbb{Z}^n$ is any full rank lattice, then there exists a Minimally Supported Frequency, Multiresolution Analysis, Composite Dilation Wavelet generating an orthonormal basis for $L^2(\mathbb{R}^n)$.*

Proof. From theorem 2.2, Γ is B -admissible. By theorem 2.9, $a = 2I_n$ is (B, Γ) -admissible. Therefore, by theorem 2.1 there exists an MSF composite dilation wavelet generating an orthonormal basis for $L^2(\mathbb{R}^n)$. \square

Example 2.1. Let us take the example we have had running through the proofs of theorems 2.2 and 2.9. Let B be the group of symmetries of the square, i.e. $B = \langle r_1, r_2 \rangle$ where r_1 is the reflection through the line $y = x$ and r_2 is the reflection through the line $y = 0$. Let $\Gamma = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} \mathbb{Z}^2$. Then $\Gamma^* = \hat{\mathbb{Z}}^2 \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{2} \end{pmatrix}$. Let $a = 2(I_n)$. Then Γ is B -admissible, a is (B, Γ) -admissible. So using theorem 2.1 we produce three wavelets to generate the space W_0 , our wavelet space. From figure 2.8, we observe that these are the wavelets ψ^1, ψ^2, ψ^3 such that

$$\hat{\psi}^1 = \sqrt{2}\chi_{[R_{-1,(1,0)} \cup R_{1,(1,0)}]}, \quad \hat{\psi}^2 = \sqrt{2}\chi_{[R_{-1,(0,1)} \cup R_{1,(0,1)}]}, \quad \hat{\psi}^3 = \sqrt{2}\chi_{[R_{-1,(1,1)} \cup R_{1,(1,1)}]}.$$

2.4 Singly Generated, MSF, Composite Dilation Wavelets

In the preceding section, theorem 2.10 tells us that we can always find an MSF composite dilation wavelet for $L^2(\mathbb{R}^n)$. In [10] and [9], it is demonstrated that in the case of MRA composite dilation wavelets, when the expanding matrix has integer determinant the number of wavelet generating functions is the size of this determinant minus 1. In Definition 1.3, this corresponds to $L = |\det(a)| - 1$.

In section 2.3 we have shown that there exists a composite dilation wavelet with 2^n wavelet generators. Obviously, as n grows, this would no longer be useful in applications. Even in dimension three, we already need 7 wavelet generating functions. However, theorem 2.10 was quite general. We could pick almost any finite group acting on \mathbb{R}^n and any full rank lattice. If we make certain choices, such as picking

the lattice according to the group, we may reduce the number of wavelet generators. Also, we generated MRA wavelets. If we are willing to surrender the MRA structure, we may reduce the number of generators.

This section provides MSF composite dilation wavelets with a single wavelet generating function. The first subsection shows that a singly generated, MRA, MSF, composite dilation wavelet exists for $L^2(\mathbb{R}^n)$ for all $n \in \mathbb{N}$. This requires us to give up the freedom to choose the group or lattice. We may keep the freedom of choosing our group if we are willing to give up the MRA structure as seen in the second part of this section. The final portion of this section demonstrates that we may keep both the MRA structure and some freedom in our choice of group when we restrict our attention to two dimensions.

2.4.1 Singly Generated MRA, MSF, Composite Dilation

Wavelets for $L^2(\mathbb{R}^n)$

Here we pick a specific group, a specific full rank lattice, and a specific expanding matrix in order to generate a singly generated MRA, MSF, composite dilation wavelet. We want to generate an orthonormal basis for $L^2(\mathbb{R}^n)$. In order to let the dimension be arbitrary, we give up the freedom of the previous section.

Theorem 2.11. *There exists a singly generated, MRA, MSF, composite dilation wavelet for $L^2(\mathbb{R}^n)$.*

Proof. First, we take the lattice to be \mathbb{Z}^n . Therefore, $\Gamma^* = \Gamma = \mathbb{Z}^n$ has the standard basis vectors $\{\hat{e}_i\}_{i=1}^n = \{e_i\}_{i=1}^n$. We let H_i be the hyperplane perpendicular to \hat{e}_i , i.e. $H_i = \hat{e}_i^\perp$. Let B be the group generated by reflections through H_i for every $i = 1, \dots, n$. We already know that \mathbb{Z}^n is B -admissible by theorem 2.2. Since \hat{e}_i lies in a hyperplane bounding the fundamental region, this is case 3 in the proof of

theorem 2.2. Therefore, we define

$$R = \left\{ \sum_{i=1}^n t_i \hat{e}_i : t_i \in [0, 1] \right\}.$$

Then R is clearly a $\hat{\mathbb{Z}}^n$ tiling set for $\hat{\mathbb{R}}^n$. We let

$$S = \bigcup_{b \in B} Rb = \left\{ \sum_{i=1}^n s_i \hat{e}_i : s_i \in [-1, 1] \right\}.$$

Then S is clearly a starlike neighborhood of the origin and R is obviously a B -tiling set for S . Therefore, with this set R , \mathbb{Z}^n is B -admissible.

Define a as the modified permutation matrix sending ξ_1 to ξ_n and ξ_i to ξ_{i-1} for $i = 2, \dots, n$ with the modification that it doubles ξ_2 . So

$$a = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Then $|\det(a)| = 2$ and $a^n = 2I_n$.

Now $S = \{(\xi_1, \dots, \xi_n) : \xi_i \in [-1, 1] \text{ for } i = 1, \dots, n\}$ so

$$Sa = \{(\xi_1, \dots, \xi_n) : \xi_1 \in [-2, 2], \xi_i \in [-1, 1] \text{ for } i = 2, \dots, n\}.$$

Then $S \subset Sa$. Also,

$$\begin{aligned} Sa \setminus S &= \{(\xi_1, \dots, \xi_n) : \xi_1 \in [-2, -1], \xi_i \in [-1, 1] \text{ for } i = 2, \dots, n\} \\ &\cup \{(\xi_1, \dots, \xi_n) : \xi_1 \in [1, 2], \xi_i \in [-1, 1] \text{ for } i = 2, \dots, n\}. \end{aligned}$$

We observe that $R = \{(\xi_1, \dots, \xi_n) : \xi_i \in [0, 1]\}$ and that

$$\begin{aligned} Ra &= \{(\xi_1, \dots, \xi_n) : \xi_1 \in [0, 2], \xi_i \in [0, 1], i = 2, \dots, n\} \\ &= R \cup \{(\xi_1, \dots, \xi_n) : \xi_1 \in [1, 2], \xi_i \in [0, 1], i = 2, \dots, n\} \end{aligned}$$

Let $R_1 = Ra \setminus R$. Then $R_1 = \{(\xi_1, \dots, \xi_n) : \xi_1 \in [1, 2], \xi_i \in [0, 1], i = 2, \dots, n\} = R + \hat{e}_1$. So R_1 is a $\hat{\mathbb{Z}}^n$ tiling set for $\hat{\mathbb{R}}^n$.

Let b_j be the reflection through the hyperplane H_j . Then we have the following action of B on R_1 :

$$R_1 b_1 = \{(\xi_1, \dots, \xi_n) : \xi_1 \in [-2, -1], \xi_i \in [0, 1], i = 2, \dots, n\} \quad (2.31)$$

$$R_1 b_j = \{(\xi_1, \dots, \xi_n) : \xi_1 \in [1, 2]; \xi_i \in [0, 1], i \neq j, i = 2, \dots, n; \xi_j \in [-1, 0]\} \quad (2.32)$$

The group B , taking combinations from (2.32), gives all possible combinations of $[0, 1]$ and $[-1, 0]$ for the every $i = 2, \dots, n$. Hence, for every $i \in \{2, \dots, n\}$, ξ_i can range over $[-1, 1]$. The reflection through H_1 (2.31) gives us the rest of $Sa \setminus S$.

$$\begin{aligned} \bigcup_{b \in B} R_1 b &= \{(\xi_1, \dots, \xi_n) : \xi_1 \in [-2, -1], \xi_i \in [-1, 1] \text{ for } i = 2, \dots, n\} \\ &\quad \cup \{(\xi_1, \dots, \xi_n) : \xi_1 \in [1, 2], \xi_i \in [-1, 1] \text{ for } i = 2, \dots, n\} \\ &= Sa \setminus S. \end{aligned} \quad (2.33)$$

By (2.33), R_1 is a B -tiling set for $Sa \setminus S$. Therefore, a is (B, \mathbb{Z}^n) -admissible.

From theorem 2.1, with $\psi = \check{\chi}_{R_1}$,

$$\Psi = \{D_a^j D_b T_k \psi : j \in \mathbb{Z}, b \in B, k \in \mathbb{Z}^n\}$$

is an orthonormal, aBG-MRA, composite dilation wavelet for $L^2(\mathbb{R}^n)$. \square

Here our example is slightly different than the running example in the previous sections. Remaining in $\hat{\mathbb{R}}^2$, let $B = \langle b_1, b_2 \rangle$ be the group generated by b_1 , the reflection through $x = 0$, and b_2 , the reflection through $y = 0$. Let $\Gamma = \mathbb{Z}^2$ and $a = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$. Then figure 2.11 shows the set Sa , and we can clearly see how $S = \bigcup_{b \in B} Rb$ and $Sa \setminus S = \bigcup_{b \in B} R_1 b$. In this case, the singly generated, MRA, MSF, composite dilation wavelet is the function ψ defined by $\hat{\psi} = \chi_{R_1}$. The associated composite dilation scaling function for the MRA is the function φ defined by $\hat{\varphi} = \chi_R$.

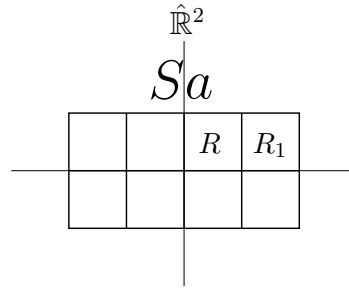


Figure 2.9: Theorem 2.11: $S = \bigcup_{b \in B} Rb$ and $Sa \setminus S = \bigcup_{B \in b} R_1 b$.

This demonstrates one of the potentially very useful properties of composite dilation wavelets. Without the dilations by the group B , we could generate a wavelet system using $\tilde{a} = 2(I_n)$ and a unit square centered at the origin. In that case, we know the wavelet space would need $|\det(a)| - 1 = 2^n - 1$ wavelet generators. However, the composite dilations allow us to transfer this need for generators into the group B . Since B is finite, B has a finite number of generators. When this number is considerably smaller than the determinant of the expanding matrix (in this section B has n generators while $|\det(\tilde{a})| = 2^n$) we actually reduce the input from $2^n - 1$ to $n + 1$ to create the orthonormal basis. If we used \tilde{a} , any implementation would require the definition of each of the $2^n - 1$ wavelet generators. Using composite dilations, an implementation would only require the n elements of the dilation group B and the single wavelet generator. It is certainly plausible that implementation algorithms can exploit the group properties to take advantage of this fact.

2.4.2 Singly Generated Non-MRA, MSF, Composite Dilation Wavelets for $L^2(\mathbb{R}^n)$

The theorem in the previous section provides MRA wavelets for any dimension at the cost of surrendering the ability to choose the group or lattice. In this section, we can retain the freedom to let the group B come from a very large family if we are willing to sacrifice the MRA structure. It is often advantageous to have the lattice be related to the group. We choose the basis for the lattice Γ^* such that each basis vector lies in a unique bounding hyperplane of the fundamental region for the group B .

Theorem 2.12. *For any group, B , with fundamental region bounded by n hyperplanes through the origin, there exists a non-MRA, MSF, composite dilation wavelet for $L^2(\mathbb{R}^n)$.*

Proof. Let B be a group with fundamental region, F , bounded by n hyperplanes through the origin. Choose a lattice Γ such that the basis vectors of Γ^* each lie in a unique hyperplane bounding F . We let R_0 be the parallelepiped formed by the basis vectors of Γ^* , $\{\gamma_i\}_{i=1}^n$:

$$Q_0 = R_0 = \left\{ \sum_{i=1}^n t_i \gamma_i : t_i \in [0, 1] \right\}.$$

Clearly, R_0 is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$. Now we take $a = 2I_n$ and for $i = 1, \dots, n$ we choose $m_i \in \{0, 1\}$ to form $\gamma = \sum_{i=1}^n m_i \gamma_i \in \Gamma^* \setminus \{0\}$. For $k \in \mathbb{N}$, we define

$$Q_k = R_0 a^{-k} + \sum_{j=0}^{k-1} \gamma a^{-j} \tag{2.34}$$

$$R_k = \left\{ \bigcup_{j=0}^k Q_j \right\} \setminus \left\{ \bigcup_{j=1}^k (Q_j - \gamma) \right\} \tag{2.35}$$

The plan is to dilate the region R_k by a^{-1} and determine the intersection. This

nonempty intersection keeps the dilations by a from being orthogonal. In each step, we remove this intersection. However, to keep the Γ^* -tiling property, we must somehow reinsert the intersection we have removed. We do that by translating the intersection by $\gamma \in \Gamma^*$. Very rapidly, these pieces become quite small, but we must take this process to its limit to ensure the orthogonality of the a dilations.

First of all, for all $k \in \mathbb{N}$,

$$\begin{aligned}
Q_k a^{-1} &= \left(R_0 a^{-k} + \sum_{j=0}^{k-1} \gamma a^{-j} \right) a^{-1} \\
&= R_0 a^{-(k+1)} + \sum_{j=0}^{k-1} \gamma a^{-(j+1)} \\
&= R_0 a^{-(k+1)} + \sum_{j=1}^k \gamma a^{-j} + \gamma - \gamma \\
&= Q_{k+1} - \gamma.
\end{aligned} \tag{2.36}$$

So, we can have an equivalent description of R_k for all $k \in \mathbb{N}$:

$$\begin{aligned}
R_k &= \left\{ \bigcup_{j=0}^k Q_j \right\} \setminus \left\{ \bigcup_{j=1}^k (Q_j - \gamma) \right\} \\
&= \left\{ \bigcup_{j=0}^k Q_j \right\} \setminus \left\{ \bigcup_{j=1}^k Q_{j-1} a^{-1} \right\} \\
&= \left\{ \bigcup_{j=0}^k Q_j \right\} \setminus \left\{ \bigcup_{j=0}^{k-1} Q_j a^{-1} \right\}.
\end{aligned} \tag{2.37}$$

Now since $a = 2I_n$, we can describe the sets Q_k explicitly.

$$\begin{aligned}
Q_k &= R_0 a^{-k} + \sum_{j=0}^{k-1} \gamma a^{-j} \\
&= \left\{ \sum_{i=1}^n t_i \gamma_i : t_i \in [0, 1] \right\} \frac{1}{2^k} + \gamma \sum_{j=0}^{k-1} \frac{1}{2^j}
\end{aligned}$$

$$\begin{aligned}
&= \left\{ \sum_{i=1}^n t_i \gamma_i : t_i \in \left[0, \frac{1}{2^k}\right] \right\} + \left(2 - \frac{1}{2^{k-1}}\right) \gamma \\
&= \left\{ \sum_{i=1}^n t_i \gamma_i : t_i \in \left[0, \frac{1}{2^k}\right] \right\} + \sum_{i=1}^n \left(2 - \frac{1}{2^{k-1}}\right) m_i \gamma_i \\
&= \left\{ \sum_{i=1}^n t_i \gamma_i : t_i \in \left[\left(2 - \frac{1}{2^{k-1}}\right) m_i, \left(2 - \frac{1}{2^{k-1}}\right) m_i + \frac{1}{2^k} \right] \right\} \tag{2.38}
\end{aligned}$$

From (2.38) and the fact that $m_i \in \{0, 1\}$, we see that when a^{-1} is applied to the sets Q_k , the resulting set is contained in R_0 :

$$\begin{aligned}
Q_k a^{-1} &= \left\{ \sum_{i=1}^n t_i \gamma_i : t_i \in \left[\left(2 - \frac{1}{2^{k-1}}\right) m_i, \left(2 - \frac{1}{2^{k-1}}\right) m_i + \frac{1}{2^k} \right] \right\} \frac{1}{2} \\
&= \left\{ \sum_{i=1}^n t_i \gamma_i : t_i \in \left[\left(1 - \frac{1}{2^k}\right) m_i, \left(1 - \frac{1}{2^k}\right) m_i + \frac{1}{2^{k+1}} \right] \right\} \\
&\subset \left\{ \sum_{i=1}^n t_i \gamma_i : t_i \in [0, 1] \right\} = R_0 \tag{2.39}
\end{aligned}$$

By (2.39) and (2.36) we have established for all $k \in \mathbb{N}$

$$Q_{k+1} \subset R_0 + \gamma \tag{2.40}$$

$$Q_k a^{-1} = Q_{k+1} - \gamma \subset R_0. \tag{2.41}$$

Using the fact that R_0 is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$, (2.40), and (2.41), we can establish that for all $k \geq 1$ and all $j \geq 0$

$$Q_k \cap Q_j a^{-1} = \emptyset. \tag{2.42}$$

From equation (2.38) we obtain that for $k \neq l$

$$Q_k \cap Q_l = \emptyset. \tag{2.43}$$

To establish (2.43), assume that $x \in Q_k \cap Q_l$ for $k \neq l$. Without loss of generality, we assume $l < k$. Since $\gamma \neq 0$ and $\gamma \in \Gamma^*$, there exist $j \in \{1, \dots, n\}$ such that $m_j \neq 0$. Since $x \in Q_l$, then (2.38) tells us that there is an $s \in [0, 1]$ such that $x_j = \left(2 - \frac{1}{2^{l-1}}\right) m_j + \frac{s}{2^l}$. Since $x \in Q_k$ then we have

$$\left(2 - \frac{1}{2^{k-1}}\right) m_j \leq \left(2 - \frac{1}{2^{l-1}}\right) m_j + \frac{s}{2^l} \leq \left(2 - \frac{1}{2^{k-1}}\right) m_j + \frac{1}{2^k}. \quad (2.44)$$

Solving (2.44) for s in the center we arrive at

$$\left(2 - \frac{1}{2^{k-l-1}}\right) m_j \leq s \leq \left(2 - \frac{1}{2^{k-l-1}}\right) m_j + \frac{1}{2^{k-l}}. \quad (2.45)$$

Since $s \in [0, 1]$ then

$$\left(2 - \frac{1}{2^{k-l-1}}\right) m_j \leq 1$$

or

$$m_j \leq \frac{1}{2 - \frac{1}{2^{k-l-1}}} = \frac{2^{k-l-1}}{2^{k-l} - 1}. \quad (2.46)$$

With $2^{k-l-1} = \frac{1}{2}2^{k-l} = (1 - \frac{1}{2})2^{k-l} = 2^{k-l} - 2^{k-l-1}$, (2.46) becomes

$$m_j \leq \frac{2^{k-l} - 2^{k-l-1}}{2^{k-l} - 1}. \quad (2.47)$$

Since $l < k$, then $2^{k-l-1} \geq 1$ and we have $m_j \leq 1$. We know $m_j \neq 0$, thus $m_j = 1$.

However, if $m_j = 1$, then (2.45) becomes

$$\left(2 - \frac{1}{2^{k-l-1}}\right) \leq s \leq \left(2 - \frac{1}{2^{k-l-1}}\right) + \frac{1}{2^{k-l}}. \quad (2.48)$$

Therefore, $1 \leq s$ and $s \in [0, 1]$. Thus, $s = 1$ and $k = l + 1$. Then the intersection can only occur on the boundary. Therefore, in the measure sense, equation (2.38) holds.

Now we want to proceed by showing that the sets $R_k \cap R_k a^{-1}$ are shrinking to a set of measure zero. We establish the following claim:

Claim 5. For all $k \geq 1$, $R_k \cap R_k a^{-1} = Q_k a^{-1}$.

Proof of claim 5: We use the alternative definition of R_k we established by (2.37).

$$R_k \cap R_k a^{-1} = \left\{ \left(\bigcup_{j=0}^k Q_j \right) \setminus \left(\bigcup_{j=0}^{k-1} Q_j a^{-1} \right) \right\} \cap \left\{ \left(\bigcup_{j=0}^k Q_j a^{-1} \right) \setminus \left(\bigcup_{j=0}^{k-1} Q_j a^{-2} \right) \right\}. \quad (2.49)$$

The left hand set of the intersection in (2.49) has already removed every $Q_j a^{-1}$ for $0 \leq j \leq k-1$. So we have

$$R_k \cap R_k a^{-1} = \left\{ \left(\bigcup_{j=0}^k Q_j \right) \setminus \left(\bigcup_{j=0}^{k-1} Q_j a^{-1} \right) \right\} \cap \left\{ Q_k a^{-1} \setminus \left(\bigcup_{j=0}^{k-1} Q_j a^{-2} \right) \right\}. \quad (2.50)$$

Using (2.42), we see $Q_k a^{-1} \cap Q_j a^{-2} = \emptyset$ for all $j \geq 0$. Then (2.50) is now

$$R_k \cap R_k a^{-1} = \left\{ \left(\bigcup_{j=0}^k Q_j \right) \setminus \left(\bigcup_{j=0}^{k-1} Q_j a^{-1} \right) \right\} \cap Q_k a^{-1}. \quad (2.51)$$

Again, we use (2.42) to see that for $j \geq 1$, $Q_j \cap Q_k a^{-1} = \emptyset$. Thus (2.51) becomes

$$\begin{aligned} R_k \cap R_k a^{-1} &= \left\{ Q_0 \setminus \left(\bigcup_{j=0}^{k-1} Q_j a^{-1} \right) \right\} \cap Q_k a^{-1} \\ &= \left\{ R_0 \setminus \left(\bigcup_{j=0}^{k-1} Q_j a^{-1} \right) \right\} \cap Q_k a^{-1}. \end{aligned} \quad (2.52)$$

But we know from (2.41) that $Q_k a^{-1} \subset R_0$ so we have reduced this intersection to

$$R_k \cap R_k a^{-1} = Q_k a^{-1} \setminus \left(\bigcup_{j=0}^{k-1} Q_j a^{-1} \right). \quad (2.53)$$

Now applying a^{-1} to equation (2.43), we see that $Q_k a^{-1} \cap Q_j a^{-1} = \emptyset$ for every $0 \leq j \leq k-1$. Hence, we have demonstrated the claim as (2.53) is now

$$R_k \cap R_k a^{-1} = Q_k a^{-1}. \quad (2.54)$$

We use claim 5 to show that the measure of these intersection is going to zero as k tends to infinity. Let m denote Lebesgue measure.

Claim 6. $m(Q_k) = \frac{1}{2^{nk}}m(R_0)$.

Proof of claim 6: This is a simple induction argument on the sets. From the definition of Q_0 , $R_0 = Q_0$. Hence

$$m(Q_0) = m(R_0) = \frac{1}{2^0}m(R_0).$$

Assume that for all $j < k$, $m(Q_j) = \frac{1}{2^{nj}}m(R_0)$. We use (2.36) and the induction hypothesis to see

$$\begin{aligned} m(Q_k) &= m(Q_k - \gamma) \\ &= m(Q_{k-1}a^{-1}) \\ &= \frac{1}{2^n}m(Q_{k-1}) \\ &= \frac{1}{2^n} \frac{1}{2^{n(k-1)}}m(R_0) \\ &= \frac{1}{2^{nk}}m(R_0) \end{aligned}$$

So the claim is valid.

Therefore, claims 5 and 6 show $\lim_{k \rightarrow \infty} m(R_k \cap R_k a^{-1}) = \lim_{k \rightarrow \infty} m(Q_k a^{-1}) = 0$.

Define $R_\infty = \lim_{k \rightarrow \infty} R_k$. Then $R_\infty \cap R_\infty a^{-1} = \emptyset$. Thus dilations by a of R_∞ are orthogonal. Also, if R_k was a Γ^* -tiling set for all $k \in \mathbb{N}$, then R_∞ is also a Γ^* -tiling set for $\hat{\mathbb{R}}^n$. We removed the intersections to keep orthogonality, but we added a Γ^* translate of the intersection back in to maintain the tiling property. We see this in the following straightforward claim.

Claim 7. For all $k \in \mathbb{N}$, R_k is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$.

Proof of claim 7: We proceed by induction on the sets R_k . First of we already know that R_0 is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$. Assume that for all $j < k + 1$, R_j is a Γ^* -tiling

set for $\hat{\mathbb{R}}^n$.

$$\begin{aligned}
R_{k+1} &= \left(\bigcup_{j=0}^{k+1} Q_j \right) \setminus \left(\bigcup_{j=0}^k Q_j a^{-1} \right) \\
&= \left\{ \left(\bigcup_{j=0}^k Q_j \right) \setminus \left(\bigcup_{j=0}^k Q_j a^{-1} \right) \right\} \cup \left\{ Q_{k+1} \setminus \left(\bigcup_{j=0}^k Q_j a^{-1} \right) \right\} \\
&= \left\{ \left(\bigcup_{j=0}^k Q_j \right) \setminus \left(\bigcup_{j=0}^{k-1} Q_j a^{-1} \right) \setminus Q_k a^{-1} \right\} \cup \left\{ Q_{k+1} \setminus \left(\bigcup_{j=0}^k Q_j a^{-1} \right) \right\} \\
&= \{ R_k \setminus Q_k a^{-1} \} \cup Q_{k+1} \\
&= \{ R_k \setminus Q_k a^{-1} \} \cup (Q_k a^{-1} + \gamma). \tag{2.55}
\end{aligned}$$

Let $A = Q_k a^{-1}$ and $B = R_k \setminus Q_k a^{-1} = R_k \setminus A$. We know from (2.36) and claim 5 that $R_k \cap R_k a^{-1} = Q_k a^{-1}$. So $A \subset R_k$. Hence $R_k = A \cup (R_k \setminus A) = A \cup B$. By the induction hypothesis, R_k is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$ and by (2.55) $R_{k+1} = (A + \gamma) \cup B$. Then lemma 2.2 establishes that R_{k+1} is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$. Hence, the claim is true.

Since R_k is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$ for all k , then $R_\infty = \lim_{k \rightarrow \infty} R_k$ is also a Γ^* -tiling set for $\hat{\mathbb{R}}^n$. This, along with the fact that the a dilates of R_∞ are orthogonal, allows us to define a orthonormal basis for $L^2(R_\infty)$, namely

$$\left\{ |\det(c)|^{\frac{1}{2}} M_k \chi_{R_\infty}(\xi) : k \in \Gamma \right\} = \left\{ |\det(c)|^{\frac{1}{2}} e^{-2\pi i \xi k} \chi_{R_\infty}(\xi) : k \in \Gamma \right\}. \tag{2.56}$$

We observe that $F = \bigcup_{j \in \mathbb{Z}} R_\infty a^j$. Therefore

$$\left\{ |\det(c)|^{\frac{1}{2}} \hat{D}_a^j M_k \chi_{R_\infty}(\xi) : j \in \mathbb{Z}, k \in \Gamma \right\} \tag{2.57}$$

is an orthonormal basis for the fundamental region, F . Since F is a fundamental

region for B , then by definition F is a B -tiling set for $\hat{\mathbb{R}}^n$ and

$$\hat{\mathbb{R}}^n = \bigcup_{b \in B} Fb.$$

Therefore, since a is diagonal,

$$\hat{\mathbb{R}}^n = \bigcup_{\substack{b \in B \\ j \in \mathbb{Z}}} R_\infty a^j b = \bigcup_{\substack{b \in B \\ j \in \mathbb{Z}}} R_\infty b a^j. \quad (2.58)$$

Using (2.57) and (2.58) we establish an orthonormal basis for $L^2(\hat{\mathbb{R}}^n)$, namely

$$\left\{ |\det(c)|^{\frac{1}{2}} \hat{D}_a^j \hat{D}_b M_k \chi_{R_\infty}(\xi) : j \in \mathbb{Z}, b \in B, k \in \Gamma \right\}. \quad (2.59)$$

Hence, taking the inverse Fourier transform of (2.59) we establish an orthonormal basis for $L^2(\mathbb{R}^n)$. Therefore, with $\psi = |\det(c)|^{\frac{1}{2}} \check{\chi}_{R_\infty}$,

$$\Psi = \left\{ D_a^j D_b T_k \psi : j \in \mathbb{Z}, b \in B, k \in \Gamma \right\} \quad (2.60)$$

is a non-MRA, composite dilation wavelet for $L^2(\mathbb{R}^n)$. \square

Example 2.2. Let $B = \langle r_1, r_2 \rangle$ be the group generated by the reflections through $y = x$ and $y = 0$. Choose the full rank lattice, $\Gamma = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mathbb{Z}^2$, so that the dual lattice has basis elements in the lines bounding the fundamental region. As in the proof, let $a = 2(I_2)$. Figure 2.10 depicts the set R_∞ when we take $\gamma = \gamma_1 + \gamma_2$. Figure 2.11 depicts the set R_∞ when $\gamma = \gamma_2$. It is apparent in each figure how the sets R_k are formed. In both cases, the wavelet ψ is defined by $\hat{\psi} = \chi_{R_\infty}$.

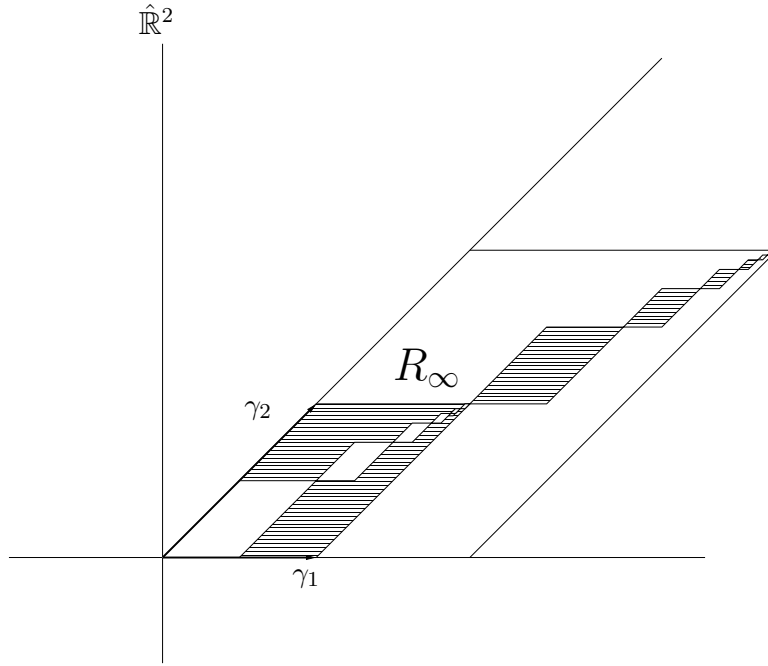


Figure 2.10: $R_\infty = \lim_{k \rightarrow \infty} R_k$ for $\gamma = \gamma_1 + \gamma_2$.

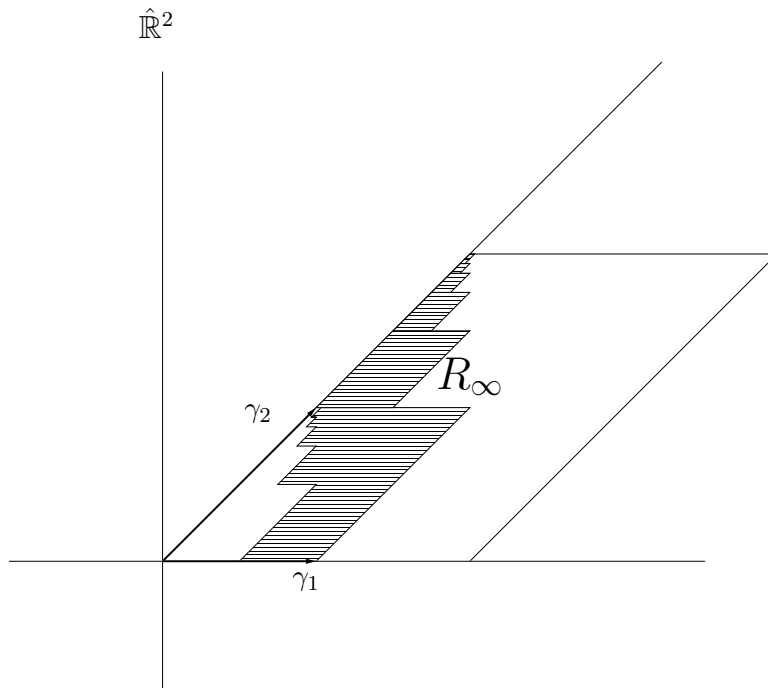


Figure 2.11: $R_\infty = \lim_{k \rightarrow \infty} R_k$ for $\gamma = \gamma_2$.

2.4.3 Singly generated, MRA, MSF, Composite Dilation Wavelets for $L^2(\mathbb{R}^2)$

The two previous subsections dealt with singly generated MSF composite dilation wavelets for $L^2(\mathbb{R}^n)$. Letting the dimension be arbitrary forced us to give something up. If we restrict ourselves to two dimensions, we can maintain both the MRA structure and the freedom in choosing our group B .

Theorem 2.13. *If B is a finite Coxeter group generated by reflections through two lines through the origin in \mathbb{R}^2 , then there exists a singly generated, MRA, MSF composite dilation wavelet for $L^2(\mathbb{R}^2)$.*

Proof. Let B be a finite Coxeter group acting on \mathbb{R}^2 with fundamental region F bounded by two lines through the origin, l_1 and l_2 . Then $B = \langle b_1, b_2 \rangle$ where b_j is the reflection through l_j for $j = 1, 2$. Then $b_1 b_2$ is a rotation that generates the subgroup $H = \langle b_1 b_2 \rangle$. Since B is finite, H is finite and $[B : H] = 2$. Also, we know that H has a fundamental region $G = F \cup F b_2$. Now a finite rotation group in \mathbb{R}^2 is generated by a rotation of $\frac{2\pi}{|H|}$. Since l_2 must bisect this angle, we know the angle between l_1 and l_2 , call it $\angle l_1 l_2$, must be $\frac{2\pi}{2|H|} = \frac{\pi}{|H|}$. Let $|H| = n \in \mathbb{N}$. Thus the angle between l_1 and l_2 is $\frac{\pi}{n}$ for some $n \in \mathbb{N}$.

Choose Γ such that the basis vectors for Γ^* , $\{\gamma_1, \gamma_2\}$, lie in l_1 and l_2 , respectively. Furthermore, we choose this basis so that $|\gamma_1| = 1, |\gamma_2| = \sqrt{2}$. As in the proof of theorem 2.2, we let

$$R = \{t_1 \gamma_1 + t_2 \gamma_2 : t_1, t_2 \in [0, 1]\}. \quad (2.61)$$

Then by the same theorem, Γ is B -admissible with $S = \bigcup_{b \in B} Rb$ a starlike neighborhood of the origin.

We denote the counter-clockwise rotation by an angle of $\frac{\pi}{n}$ by $\rho(\frac{\pi}{n})$. Let $a = \sqrt{2}\rho(\frac{\pi}{n})$. Recall that in the frequency domain, the dilations act on the right. Therefore

$a = \sqrt{2}\rho(\frac{\pi}{n}) = \sqrt{2} \begin{pmatrix} \cos(\frac{\pi}{n}) & \sin(\frac{\pi}{n}) \\ -\sin(\frac{\pi}{n}) & \cos(\frac{\pi}{n}) \end{pmatrix}$. Then a is an expanding matrix. Since $\angle l_1 l_2 = \frac{\pi}{n}$ and $|\gamma_1| = 1, |\gamma_2| = \sqrt{2}$, then $\gamma_1 a = \gamma_2$ and $\gamma_2 a = (2\gamma_1) b_2$. Therefore, since $\gamma_2 b_2 = \gamma_2$, we have

$$\begin{aligned}
Ra &= \{t_1 \gamma_1 a + t_2 \gamma_2 a : t_1, t_2 \in [0, 1]\} \\
&= \{t_1 \gamma_2 + 2t_2 \gamma_1 b_2 : t_1, t_2 \in [0, 1]\} \\
&= \{t_1 \gamma_1 b_2 + t_2 \gamma_2 : t_1, t_2 \in [0, 1]\} \cup (\{t_1 \gamma_1 b_2 + t_2 \gamma_2 : t_1, t_2 \in [0, 1]\} + \gamma_1 b_2) \\
&= Rb_2 \cup (Rb_2 + \gamma_1 b_2)
\end{aligned} \tag{2.62}$$

Applying $b_2 = b_2^{-1}$ to (2.62) we have

$$Rab_2 = R \cup (R + \gamma_1). \tag{2.63}$$

Figure 2.12 depicts equation (2.63) for a specific example. Let B be the group generated by b_1 , the reflection through $y = 0$, and b_2 , the reflection through $y = x$. Then B is also generated by the rotation $\rho(\frac{\pi}{4}) = b_1 b_2$ and b_2 ; $B = \langle b_1, b_2 \rangle = \langle b_1 b_2, b_2 \rangle$.

$$\Gamma = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mathbb{Z}^2 \text{ and } a = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \sqrt{2}\rho(\frac{\pi}{4}).$$

Resuming the proof, we look at the B orbit of Ra :

$$\begin{aligned}
\bigcup_{b \in B} Rab &= \bigcup_{b \in B} Rab_2 b \\
&= \bigcup_{b \in B} [R \cup (R + \gamma_1)] b \\
&= \left(\bigcup_{b \in B} Rb \right) \cup \left[\bigcup_{b \in B} (Rb + \gamma_1 b) \right] \\
&= S \cup \left[\bigcup_{b \in B} (R + \gamma_1) b \right].
\end{aligned} \tag{2.64}$$

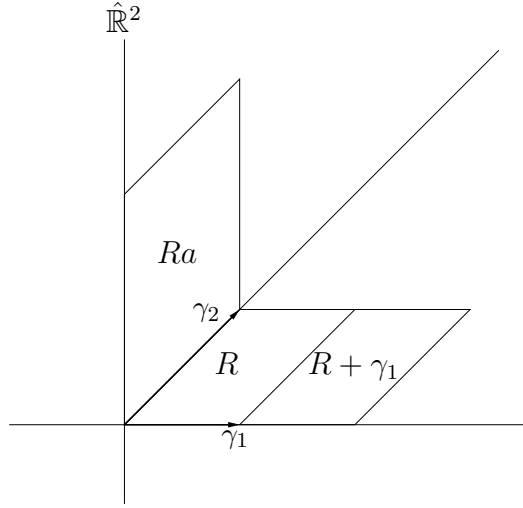


Figure 2.12: $Rab_2 = R \cup (R + \gamma_1)$.

In \mathbb{R}^2 , all rotations commute since $\rho(\theta)\rho(\phi) = \rho(\theta + \phi) = \rho(\phi)\rho(\theta)$. Also when we fix a line in \mathbb{R}^2 , call it l , then the reflection through the line l , call it r_l , has a commuting relationship with all rotations. This relationship is

$$\rho(\theta)r_l = r_l[\rho(\theta)]^{-1} = r_l\rho(-\theta). \quad (2.65)$$

Therefore, we can easily see that for any angle θ

$$\rho(\theta)r_l[\rho(\theta)]^{-1} = \rho(\theta)\rho(\theta)r_l = \rho(2\theta)r_l. \quad (2.66)$$

In our situation we had fixed the lines l_1, l_2 in \mathbb{R}^2 and defined $b_1 = r_{l_1}, b_2 = r_{l_2}$ as the reflections through these lines. We also know that $B = \langle b_1, b_2 \rangle = \langle b_1b_2, b_2 \rangle$. From the previous discussion, we have $b_1b_2 = \rho(\frac{2\pi}{n})$ since $\angle l_1l_2 = \frac{\pi}{n}$. Also, $a = \sqrt{2}\rho(\frac{\pi}{n})$. Therefore,

$$\begin{aligned} a(b_1b_2)a^{-1} &= \sqrt{2}\rho\left(\frac{\pi}{n}\right)\rho\left(\frac{2\pi}{n}\right)\rho\left(-\frac{\pi}{n}\right)\frac{1}{\sqrt{2}} \\ &= \rho\left(\frac{2\pi}{n}\right) \end{aligned}$$

$$= b_1 b_2 \quad (2.67)$$

$$\begin{aligned} ab_2 a^{-1} &= \sqrt{2} \rho\left(\frac{\pi}{n}\right) r_{l_2} \rho\left(-\frac{\pi}{n}\right) \frac{1}{\sqrt{2}} \\ &= \rho\left(\frac{2\pi}{n}\right) r_{l_2} \\ &= (b_1 b_2) b_2 \\ &= b_1 \end{aligned} \quad (2.68)$$

Since $b_1 b_2$ and b_2 generate B , then by (2.67) and (2.68) we have $aBa^{-1} = B$. Hence

$$Sa = \left(\bigcup_{b \in B} Rb \right) a = \bigcup_{b \in B} Rba = \bigcup_{b \in B} Rab. \quad (2.69)$$

Thus, combining (2.64) and (2.69) and defining $R_1 = R + \gamma_1$, we observe that $S \subset Sa$ and

$$Sa \setminus S = \bigcup_{b \in B} R_1 b. \quad (2.70)$$

Since R was a Γ^* -tiling set for $\hat{\mathbb{R}}^n$, $\gamma_1 \in \Gamma^*$, and $R_1 = R + \gamma_1$, the lemma 2.2 provides that R_1 is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$. Equation (2.70) provides that R_1 is a B -tiling set for $Sa \setminus S$. Therefore, $a = \sqrt{2} \rho(\frac{\pi}{n})$ is (B, Γ) -admissible. So by theorem 2.1, for $\psi = |\det(c)|^{\frac{1}{2}} \tilde{\chi}_{R_1}$, we have

$$\Psi = \{D_a^j D_b T_k \psi : j \in \mathbb{Z}, b \in B, k \in \Gamma\}$$

is a singly generated, orthonormal, MRA, MSF, composite dilation wavelet for $L^2(\mathbb{R}^2)$.

□

Figure 2.13 depicts the sets $Sa, Sa \setminus S, S, R$, and R_1 for the example that ran through the proof. Let $\hat{\phi} = \chi_R$ and $\hat{\psi} = \chi_{R_1}$. $B = \langle b_1, b_2 \rangle = \langle b_1 b_2, b_2 \rangle$ is the group generated by b_1 , the reflection through $y = 0$, and b_2 , the reflection through $y = x$.

B is also generated by the rotation $\rho(\frac{\pi}{4}) = b_1 b_2$ and b_2 . $\Gamma = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mathbb{Z}^2$ and

$$a = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \sqrt{2} \rho(\frac{\pi}{4}).$$

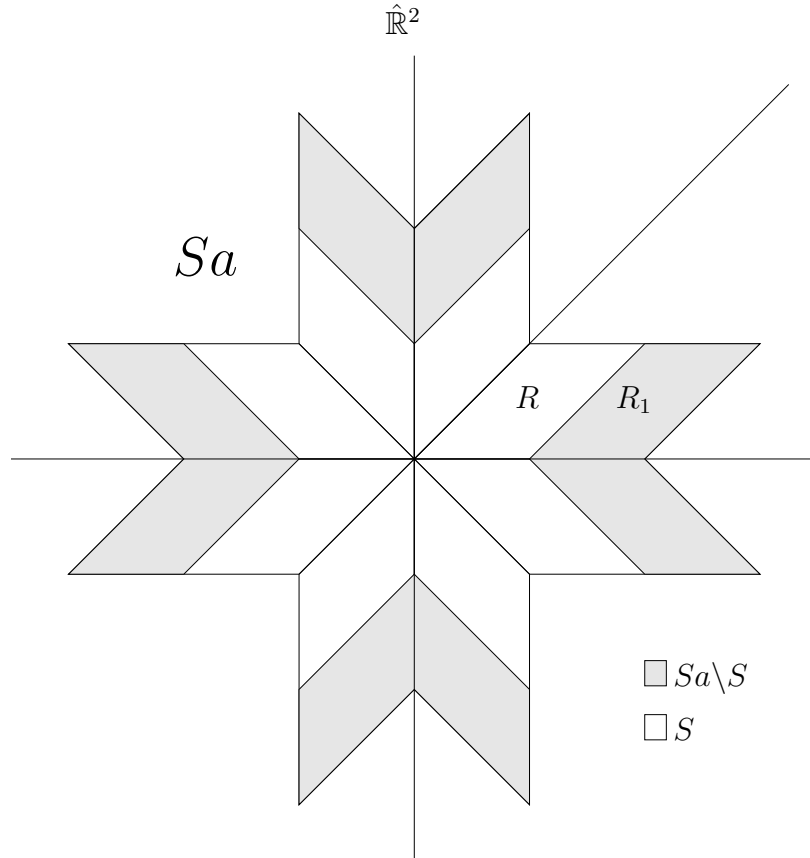


Figure 2.13: R is a B -tiling set for S . R_1 is a B -tiling set for $Sa \setminus S$.

Now we may also obtain a singly generated, MRA, MSF composite dilation wavelet with rotation groups and an expanding matrix that is a scalar multiple of an appropriate reflection.

Theorem 2.14. *If B is a finite rotation group acting on \mathbb{R}^2 , there exists a singly generated, MSF, MRA composite dilation wavelet for $L^2(\mathbb{R}^2)$.*

Proof. This proof is quite similar to the preceding proof. Let B be a rotation group,

$|B| = n \in \mathbb{N}$. A fundamental region for B is necessarily bounded by two lines through the origin, l_1 and l_2 . Here, $\angle l_1 l_2 = \frac{2\pi}{|B|} = \frac{2\pi}{n}$. Thus $B = \langle \rho(\frac{2\pi}{n}) \rangle$. Choose Γ and R as in the previous proof. Again, Γ is B -admissible with $S = \cup_{b \in B} Rb$ a starlike neighborhood of the origin.

Let $r = r_{l_2}$ be the reflection through l_2 . Define $a = \sqrt{2}r$ so that a is expanding. With $|\gamma_1| = 1$, then $\gamma_1 \rho(\frac{2\pi}{n}) = \frac{1}{\sqrt{2}}\gamma_2$. Similarly, with $|\gamma_2| = \sqrt{2}$, $\gamma_2 \rho(\frac{2\pi}{n}) = \sqrt{2}\gamma_1 r$. So, $\gamma_1 a = \sqrt{2}\gamma_1 r = \gamma_2 \rho(\frac{2\pi}{n})$ and $\gamma_2 a = \sqrt{2}\gamma_2 r = \sqrt{2}\gamma_2 = 2\gamma_1 \rho(\frac{2\pi}{n})$. Then

$$\begin{aligned}
Ra &= \{t_1 \gamma_1 a + t_2 \gamma_2 a : t_1, t_2 \in [0, 1]\} \\
&= \left\{ t_1 \gamma_2 \rho\left(\frac{2\pi}{n}\right) + t_2 \left[2\gamma_1 \rho\left(\frac{2\pi}{n}\right) \right] : t_1, t_2 \in [0, 1] \right\} \\
&= \{2t_1 \gamma_1 + t_2 \gamma_2 : t_1, t_2 \in [0, 1]\} \rho\left(\frac{2\pi}{n}\right) \\
&= [\{t_1 \gamma_1 + t_2 \gamma_2 : t_1, t_2 \in [0, 1]\} \cup (\{t_1 \gamma_1 + t_2 \gamma_2 : t_1, t_2 \in [0, 1]\} + \gamma_1)] \rho\left(\frac{2\pi}{n}\right) \\
&= [R \cup (R + \gamma_1)] \rho\left(\frac{2\pi}{n}\right). \tag{2.71}
\end{aligned}$$

Figure 2.14 depicts equation (2.71) for a specific example. Let $B = \langle \rho(\frac{\pi}{4}) \rangle$ be the rotation group generated by a rotation of $\frac{\pi}{4}$, $a = \sqrt{2}r$ where r is the reflection through $y = x$, and $\Gamma = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mathbb{Z}^2$.

We continue the proof by defining $R_1 = R + \gamma_1$. Then we have $Ra \rho(-\frac{2\pi}{n}) = R \cup R_1$. Using arguments similar to those in the previous proof, we show that $aBa^{-1} = B$:

$$\begin{aligned}
a \rho\left(\frac{2\pi}{n}\right) a^{-1} &= \sqrt{2} r_l \rho\left(\frac{2\pi}{n}\right) r_l \frac{1}{\sqrt{2}} \\
&= \rho\left(-\frac{2\pi}{n}\right) r_l r_l \\
&= \rho\left(-\frac{2\pi}{n}\right).
\end{aligned}$$

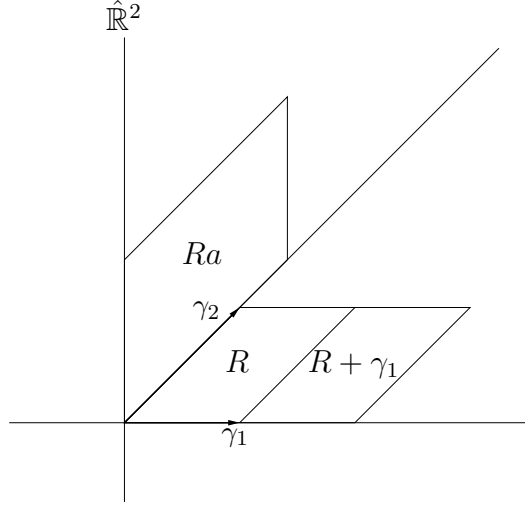


Figure 2.14: $Ra\rho\left(\frac{-\pi}{4}\right) = R \cup (R + \gamma_1)$.

Then, with $S = \cup_{b \in B} Rb$, we have

$$\begin{aligned}
Sa &= \bigcup_{b \in B} Rba \\
&= \bigcup_{b \in B} Rab \\
&= \bigcup_{b \in B} Ra\rho\left(-\frac{2\pi}{n}\right)b \\
&= \bigcup_{b \in B} [R \cup (R + \gamma_1)]b \\
&= \left\{ \bigcup_{b \in B} Rb \right\} \cup \left\{ \bigcup_{b \in B} R_1b \right\} \\
&= S \cup \left\{ \bigcup_{b \in B} R_1b \right\}.
\end{aligned}$$

Then $S \subset Sa$ and R_1 is a B -tiling set of $Sa \setminus S$. Lemma 2.2 provides that R_1 is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$. Therefore, a is (B, Γ) -admissible.

Hence, with $\psi = |\det(c)|^{\frac{1}{2}} \check{\chi}_{R_1}$,

$$\Psi = \{D_a^j D_b T_k \psi : j \in \mathbb{Z}, b \in B, k \in \Gamma\}$$

is a singly generated, orthonormal, MRA, MSF, composite dilation wavelet for $L^2(\mathbb{R}^2)$.

□

Figure 2.15 depicts the sets Sa , $Sa \setminus S$, S , R , and R_1 for the example that ran through the proof of theorem 2.14. Let $\hat{\phi} = \chi_R$ and $\hat{\psi} = \chi_{R_1}$. $B = \langle \rho(\frac{\pi}{4}) \rangle$ is the group generated by the rotation $\rho(\frac{\pi}{4})$. $\Gamma = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mathbb{Z}^2$ and $a = \sqrt{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

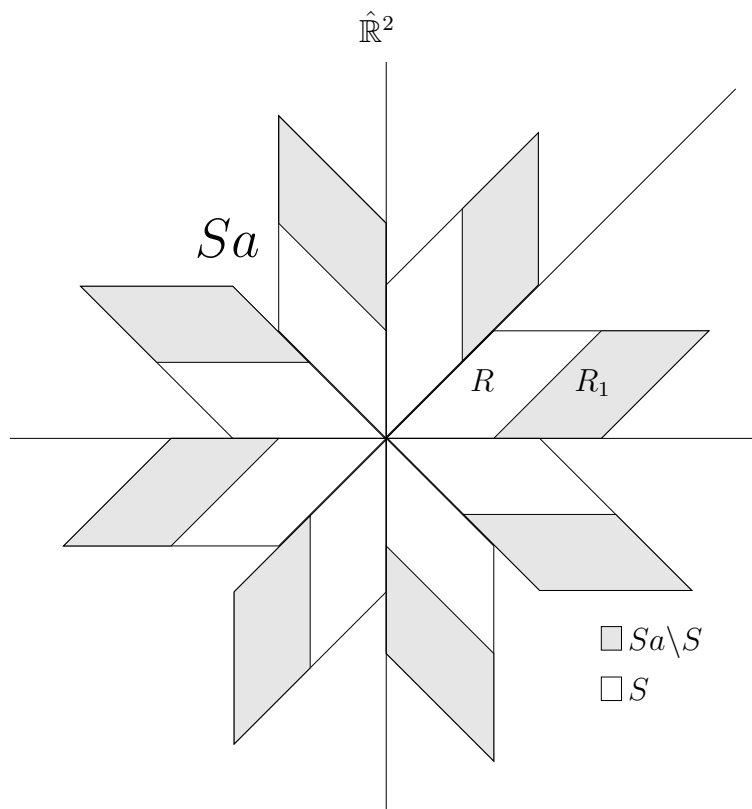


Figure 2.15: R is B -tiling set for S and R_1 is a B -tiling set for $Sa \setminus S$.

2.5 MSF, Composite Dilation, Parseval Frame Wavelets

So far, we have focused on composite dilation wavelets generating orthonormal bases for $L^2(\mathbb{R}^n)$. Relaxing the orthonormality condition can be advantageous in many ways. Frames, especially Parseval frames, are used extensively in applications as they may possess a useful redundancy with very little additional computational cost. Additionally, it is usually easier to produce a Parseval frame than to produce an orthonormal basis.

The same is true for minimally supported frequency composite dilation wavelets. Obviously, every orthonormal basis is a Parseval frame, so the existence of Parseval frame composite dilation wavelets has been established in the preceding sections. However, it becomes easier to find Parseval frame composite dilation wavelets as the orthonormality requires the characteristic functions to be supported on lattice tiling sets for $\hat{\mathbb{R}}^n$. To find MSF, composite dilation, Parseval frame wavelets, this condition is relaxed to the situation where the characteristic functions are supported on sets contained in lattice tiling sets for $\hat{\mathbb{R}}^n$. This significantly reduces the effort required to find such wavelets. We begin with versions of the admissibility conditions for Parseval frames.

Definition 2.5. A lattice $\Gamma = c\mathbb{Z}^n$ ($c \in GL_n(\mathbb{R})$) is *PF-B-admissible* if there exist a group B and a measurable set $R \subset \hat{\mathbb{R}}^n$ such that:

- (i) $R \subset W$ where W is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$ and
- (ii) there exists a starlike neighborhood of 0, S , such that R is a B -tiling set for S .

Definition 2.6. Let Γ be PF-B-admissible and let S be a starlike neighborhood of 0 satisfying (ii) from definition 2.5. A matrix $a \in GL_n(\mathbb{R})$ is *PF-(B, Γ)-admissible* if a is expanding with $S \subset Sa$ and there exist $R_1, \dots, R_L \subset Sa \setminus S$ such that:

- (i) for all $l = 1 \dots L$, $R_l \subset W_l$ where W_l is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$ and
- (ii) $\bigcup_{l=1}^L R_l$ is a B -tiling set for $Sa \setminus S$.

Because we only need to find sets contained in lattice tiling sets, these Parseval frame admissibility conditions are much easier to satisfy. In the previous sections we showed that we can satisfy them since any lattice that is B -admissible is also PF- B -admissible and any matrix that is (B, Γ) -admissible is also PF- (B, Γ) -admissible. However, we will prove analogous theorems in this section to highlight how the Parseval frame requirements are easier to satisfy.

Theorem 2.15. *If B is a finite group whose fundamental region is bounded by hyperplanes through 0, then every lattice $\Gamma = c\mathbb{Z}^n$ is PF- B -admissible.*

Proof. Suppose B is a finite group whose fundamental region is bounded by hyperplanes through 0 and $\Gamma = c\mathbb{Z}^n$. Let F be a fundamental domain of $\hat{\mathbb{R}}^n$ for the group B ; that is, F is a B -tiling set for $\hat{\mathbb{R}}^n$. Let $\{\gamma_i\}_{i=1}^n$ be a basis for Γ^* . Define $P = \{\sum_{i=1}^n t_i \gamma_i : t_i \in [0, 1]\}$. Then P is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$. If we translate P , it is still a Γ^* -tiling set for $\hat{\mathbb{R}}^n$. Therefore, translate P such that the origin is in the interior of P . Say $0 \in P - \alpha$ for some $\alpha \in \hat{\mathbb{R}}^n$. Define $R = (P - \alpha) \cap F$. Then $R \subset P - \alpha$. Also, since $P - \alpha$ and F are both starlike with respect to 0, then R is starlike with respect to 0. By lemma 2.5, $S = \bigcup_{b \in B} Rb$ is a starlike neighborhood of 0. Therefore, Γ is PF- B -admissible. \square

Again, we look at a two dimensional example. Figures 2.16 and 2.17 depict R and S for $\alpha = (\frac{1}{4}, \frac{1}{16})$, $B = \langle r_1, r_2 \rangle$, the group generated by reflections through $y = x$ and $y = 0$, and $\Gamma = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} \mathbb{Z}^2$.

For Parseval frames, we eliminated the need to replace the portion of the parallelepiped formed by the lattice basis that does not lie in F . It is also no longer

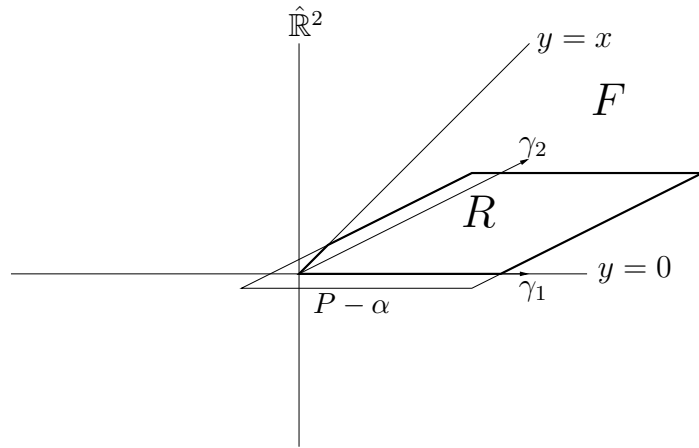


Figure 2.16: PF- B -admissible: $R = (P - \alpha) \cap F$.

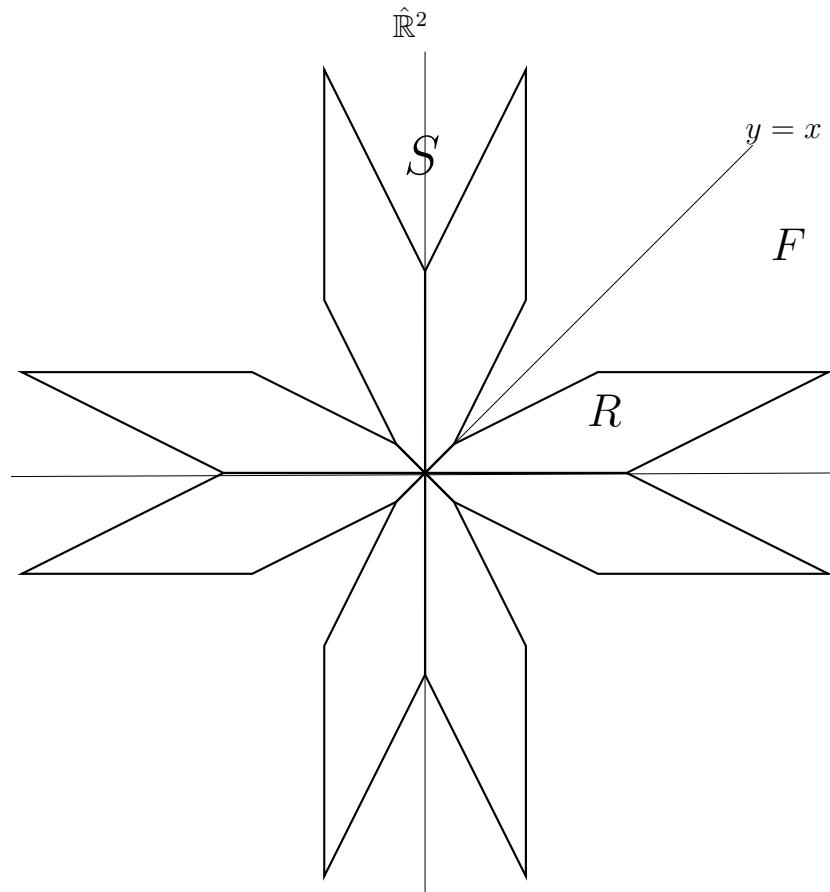


Figure 2.17: PF- B -admissible: R is B -tiling set for S .

necessary to concern ourselves with how the lattice relates to the fundamental domain for B as we no longer need to translate any of the set $(P - \alpha) \cap F^c$.

In the orthonormal case, finding an expanding matrix $a \in GL_n(\mathbb{R})$ is rather difficult. We actually only established that twice the identity matrix is always (B, Γ) -admissible. However, every expanding matrix, a , that normalizes B and satisfies $S \subset Sa$ is PF- (B, Γ) -admissible since we can simply intersect the region $Sa \setminus S$ with lattice tilings.

Theorem 2.16. *Let B be a finite group whose fundamental region is bounded by hyperplanes through 0 and $\Gamma = c\mathbb{Z}^n$. Suppose S is a starlike neighborhood of 0 satisfying (ii) from definition 2.5. If $a \in GL_n(\mathbb{R})$ is expanding such that $S \subset Sa$ and $aB = Ba$, then a is PF- (B, Γ) -admissible.*

Proof. Suppose R is any set satisfying the conditions of definition 2.5. Suppose $S = \bigcup_{b \in B} Rb$ is a starlike neighborhood of the origin and $a \in GL_n(\mathbb{R})$ is expanding such that $S \subset Sa$ and $aB = Ba$. Define $Q = Ra \setminus S$. Then

$$\begin{aligned}
\bigcup_{b \in B} Qb &= \bigcup_{b \in B} (Ra \setminus S) \\
&= \left\{ \bigcup_{b \in B} Rab \right\} \setminus \left\{ \bigcup_{b \in B} Sb \right\} \\
&= \left\{ \bigcup_{\tilde{b} \in B} R\tilde{b}a \right\} \setminus S \\
&= \left(\bigcup_{b \in B} Rb \right) a \setminus S \\
&= Sa \setminus S.
\end{aligned} \tag{2.72}$$

Therefore Q is a B -tiling set of $Sa \setminus S$.

Let P be the parallelepiped formed by taking the convex hull of the lattice basis for Γ^* as in the proof of theorem 2.15. Then P is a Γ^* -tiling set of $\hat{\mathbb{R}}^n$. Since Q is compact,

there exist a smallest integer L and $\alpha_1, \dots, \alpha_L \in \Gamma^*$ such that $Q \subset \bigcup_{l=1}^L (P + \alpha_l)$. Define $R_l = Q \cap (P + \alpha_l)$. Then $R_l \subset (P + \alpha_l)$ and $(P + \alpha_l)$ is a Γ^* -tiling set for $\hat{\mathbb{R}}^n$. Furthermore, by (2.72), $Q = \bigcup_{l=1}^L R_l$ is a B -tiling set of $Sa \setminus S$.

Therefore, a is PF- (B, Γ) -admissible. \square

Continuing with the example from figures 2.16 and 2.17, we let $\alpha = (\frac{1}{4}, \frac{1}{16})$, $B = \langle r_1, r_2 \rangle$, the group generated by reflections through $y = x$ and $y = 0$, and $\Gamma = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} \mathbb{Z}^2$. The for $a = \frac{3}{2}(I_2)$, figure 2.18 shows Sa and how R and R_1 are B -tiling sets for S and $Sa \setminus S$, respectively.

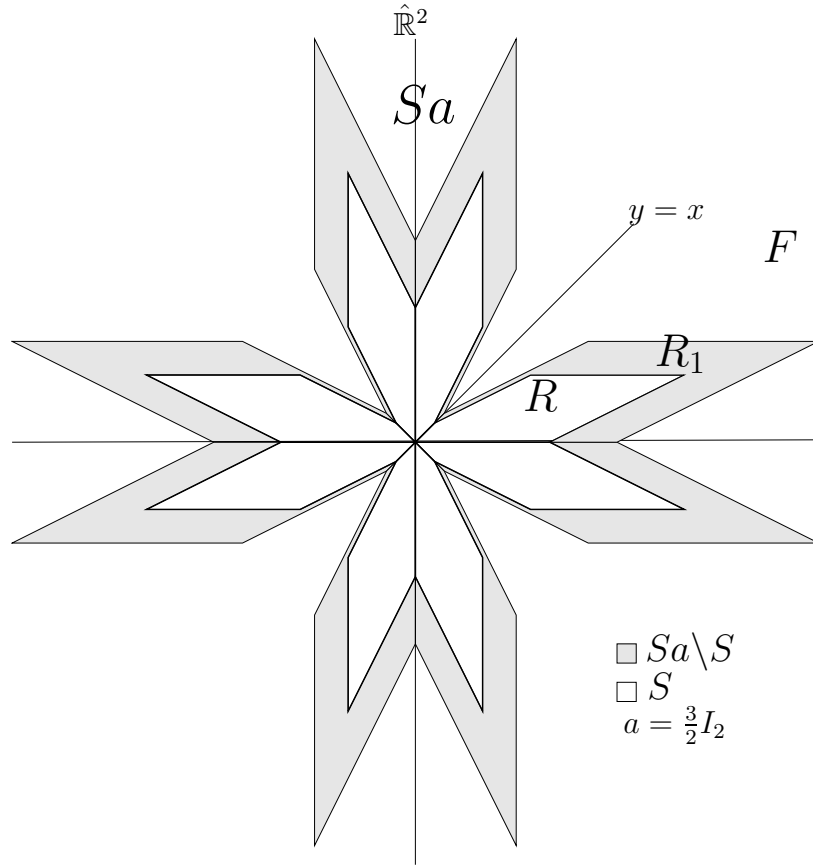


Figure 2.18: PF- (B, Γ) -admissible: R is B -tiling set for S , R_1 is a B -tiling set for $Sa \setminus S$.

Again, it is important to have admissibility conditions that actually produce wavelets. We will establish that given a, B, Γ satisfying the Parseval frame admissibility conditions, we are able to generate an MSF, composite dilation, Parseval frame wavelet for $L^2(\mathbb{R}^n)$. First, we cover a few elementary lemmas that will be useful.

Lemma 2.17. *If, for each $l = 1, \dots, L$, $\{x_i^l\}_{i=1}^\infty$ is a Parseval frame for $L^2(\mathbb{R}^n)$, then $\bigcup_{l=1}^L \{\frac{1}{L}x_i^l\}_{i=1}^\infty$ is a Parseval frame for $L^2(\mathbb{R}^n)$.*

Proof. Let $f \in L^2(\mathbb{R}^n)$. For each l , $\sum_{i=1}^\infty |\langle f, x_i^l \rangle|^2 = \|f\|_2^2$. Therefore

$$\begin{aligned} \sum_{l=1}^L \sum_{i=1}^\infty \left| \left\langle f, \frac{1}{L}x_i^l \right\rangle \right|^2 &= \frac{1}{L} \sum_{l=1}^L \sum_{i=1}^\infty |\langle f, x_i^l \rangle|^2 \\ &= \frac{1}{L} \sum_{l=1}^L \|f\|_2^2 \\ &= \frac{1}{L} L \|f\|_2^2 \\ &= \|f\|_2^2. \end{aligned}$$

Hence, $\bigcup_{l=1}^L \{\frac{1}{L}x_i^l\}_{i=1}^\infty$ is a Parseval frame for $L^2(\mathbb{R}^n)$. \square

For the remaining discussion in this section, we simplify notation by defining $e_k(\xi) = |\det(c)|^{\frac{1}{2}} M_k \chi_R(\xi) = |\det(c)|^{\frac{1}{2}} e^{-2\pi i \xi k} \chi_R(\xi)$ where $c \in GL_n(\mathbb{R})$ and R are obvious from context.

Lemma 2.18. *Let $\Gamma = c\mathbb{Z}^n$. If W is a Γ^* -tiling set of $\hat{\mathbb{R}}^n$ and $R \subset W$, then $\{e_k : k \in \Gamma\}$ is a Parseval frame for $L^2(R)$.*

Proof. We simply establish the Parseval identity. Let $f \in L^2(\hat{\mathbb{R}}^n)$. Since $R \subset W$, $\|f\|_{L^2(R)} = \|f\chi_R\|_{L^2(W)}$. We know that the set $\{|\det(c)|^{\frac{1}{2}} M_k \chi_W : k \in \Gamma\}$ is an orthonormal basis for $L^2(W)$ and thus satisfies the Parseval identity. Hence,

$$\|f\|_{L^2(R)}^2 = \|f\chi_R\|_{L^2(W)}^2$$

$$\begin{aligned}
&= \sum_{k \in \Gamma} \left| \left\langle f \chi_R, |\det(c)|^{\frac{1}{2}} e^{-2\pi i(\cdot)k} \chi_W \right\rangle \right|^2 \\
&= \sum_{k \in \Gamma} \left| \left\langle f, |\det(c)|^{\frac{1}{2}} e^{-2\pi i(\cdot)k} \chi_W \chi_R \right\rangle \right|^2 \\
&= \sum_{k \in \Gamma} \left| \left\langle f, |\det(c)|^{\frac{1}{2}} e^{-2\pi i(\cdot)k} \chi_R \right\rangle \right|^2 \\
&= \sum_{k \in \Gamma} |\langle f, e_k \rangle|^2
\end{aligned}$$

Therefore, $\{e_k : k \in \Gamma\}$ is a Parseval frame for $L^2(R)$. \square

Lemma 2.19. *Let $\Gamma = c\mathbb{Z}^n$. If W is a Γ^* -tiling set of $\hat{\mathbb{R}}^n$, $R \subset W$, and R is a B -tiling set for S , then $\{\hat{D}_b e_k : b \in B, k \in \Gamma\}$ is a Parseval frame for $L^2(S)$.*

Proof. We know that $\{e_k : k \in \Gamma\}$ is a Parseval frame for $L^2(R)$. Let $f \in L^2(S)$.

Then

$$\begin{aligned}
\sum_{b \in B} \sum_{k \in \Gamma} \left| \left\langle f, \hat{D}_{b^{-1}} e_k \right\rangle \right|^2 &= \sum_{b \in B} \sum_{k \in \Gamma} \left| \left\langle \hat{D}_b f, e_k \right\rangle \right|^2 \\
&= \sum_{b \in B} \left\| \hat{D}_b f \right\|_{L^2(R)}^2 \\
&= \sum_{b \in B} \int_R \left| \hat{D}_b f(\xi) \right|^2 d\xi \\
&= \sum_{b \in B} \int_R |f(\xi b)|^2 d\xi \\
&= \sum_{b \in B} \int_{Rb} |f(\xi)|^2 d\xi \\
&= \int_{S = \bigcup_{b \in B} Rb} |f(\xi)|^2 d\xi \\
&= \|f\|_{L^2(S)}^2.
\end{aligned}$$

Therefore, $\{\hat{D}_b e_k : b \in B, k \in \Gamma\}$ satisfies the Parseval identity and is a Parseval frame for $L^2(S)$. \square

Now we are ready to establish that with a very large family of groups B we can

generate MRA, MSF, composite dilation Parseval frame wavelets for $L^2(\mathbb{R}^n)$. The following theorem uses the preceding lemmas and the standard MRA arguments.

Theorem 2.20. *Suppose B is a finite group with fundamental domain bounded by hyperplanes through the origin, $\Gamma = c\mathbb{Z}^n$ is PF- B -admissible with R and S satisfying definition 2.5, and $a \in GL_n(\mathbb{R})$ is PF- (B, Γ) -admissible with R_1, \dots, R_L satisfying definition 2.6. Then, for $\psi^l = \frac{1}{L} |\det(c)|^{\frac{1}{2}} \check{\chi}_{R_l}$,*

$$\{D_a^j D_b T_k \psi^l : j \in \mathbb{Z}, b \in B, k \in \Gamma, l = 1, \dots, L\}$$

is an MRA, MSF, composite dilation Parseval frame wavelet.

Proof. An MRA, Parseval frame wavelet has a scaling function φ and wavelets ψ^l for $l = 1, \dots, L$ that generate Parseval frames for the spaces V_0 and W_0 respectively. Define $V_0 = \check{L}^2(S)$ and $V_j = D_a^{-j} V_0 = \check{L}^2(Sa^j)$. Since $S \subset Sa$, then $\{V_j\}_{j=-\infty}^{\infty}$ is a nested sequence with $V_j \subset V_{j+1}$.

Define $\varphi(x) = |\det(c)|^{\frac{1}{2}} \check{\chi}_R(x) = \check{e}_0(\xi)$. Then $[T_k \varphi]^\wedge(\xi) = e_k(\xi)$. Since R is a B -tiling set for S , lemma 2.19 tells us that $\{\hat{D}_b [T_k \varphi]^\wedge : b \in B, k \in \Gamma\}$ is a Parseval frame for $L^2(S)$. Thus,

$$\{D_b T_k \varphi : b \in B, k \in \Gamma\}$$

is a Parseval frame for V_0 .

Suppose $L = 1$ so $Sa \setminus S = \bigcup_{b \in B} R_1 b$ and R_1 is contained in a Γ^* -tiling set for $\hat{\mathbb{R}}^n$. Then, with $\psi = |\det(c)|^{\frac{1}{2}} \check{\chi}_{R_1}$, a similar argument provides that

$$\{D_b T_k \psi : b \in B, k \in \Gamma\}$$

is a Parseval frame for $L^2(Sa \setminus S)$. Define $W_0 = \check{L}^2(Sa \setminus S)$. Then $W_0 = V_1 \cap V_0^\perp$.

The argument demonstrating that this is an MRA follows the argument in the proof of theorem 2.1. Since φ generates a Parseval frame for V_0 and ψ generates a

Parseval frame for W_0 ,

$$\{D_a^j D_b T_k \psi : j \in \mathbb{Z}, b \in B, k \in \Gamma\}$$

is an MSF, MRA, composite dilation Parseval frame wavelet.

If $L > 1$, then with $\psi^l = \frac{1}{L} |\det(c)|^{\frac{1}{2}} \chi_{R_l}$, lemma 2.17 provides that

$$\{D_a^j D_b T_k \psi^l : j \in \mathbb{Z}, b \in B, k \in \Gamma, l = 1, \dots, L\}$$

is an MRA, MSF, composite dilation Parseval frame wavelet for $L^2(\mathbb{R}^n)$. \square

The relaxation of the lattice tiling property creates a situation where Parseval frame wavelets with the MRA structure are much easier to find. In the preceding section, we discussed singly generated MSF, composite dilation wavelets. When we sacrificed the MRA structure, we could achieve the desired single generator for every group with the appropriate fundamental domain. If we want to keep the MRA structure and are willing to accept Parseval frames in lieu of the orthonormal basis, we may obtain singly generated, MSF, MRA, composite dilation Parseval frame wavelets.

Theorem 2.21. *For any group, B , with fundamental region bounded by n hyperplanes through the origin, there exists a singly generated, MRA, MSF, composite dilation Parseval frame wavelet for $L^2(\mathbb{R}^n)$.*

Proof. Recall the proof of theorem 2.12. Since we only need characteristic function supported on a subset of a lattice tiling set, we can simply take the set $R_0 a^{-1}$ as the support of our scaling function. Let's see this explicitly.

Let B be a group with fundamental region, F , bounded by n hyperplanes through the origin. Choose a lattice $\Gamma = c\mathbb{Z}^n$ such that the basis vectors of Γ^* each lie in a unique hyperplane bounding F . We let P be the parallelepiped formed by the basis

vectors of Γ^* , $\{\gamma_i\}_{i=1}^n$:

$$P = \left\{ \sum_{i=1}^n t_i \gamma_i : t_i \in [0, 1] \right\}.$$

Then P is clearly a Γ^* -tiling set for $\hat{\mathbb{R}}^n$. Now take $a = 2I_n$. Define $R = Pa^{-1}$ so

$$R = \left\{ \sum_{i=1}^n s_i \gamma_i : s_i \in \left[0, \frac{1}{2}\right] \right\}.$$

Then $R \subset P$. By definition, R is a starlike with respect to the origin. Define $S = \bigcup_{b \in B} Rb$. Then R is a B -tiling set for S and S is a starlike neighborhood of the origin. Hence, with R and S as defined, Γ is PF- B -admissible.

The matrix $a = 2I_n$ is expanding, $S \subset Sa$, and $aB = Ba$. By theorem 2.16, a is PF- (B, Γ) -admissible. For clarity, we describe the sets explicitly. Since $R = Pa^{-1}$, then $Ra = P$. Let $Q = Ra \setminus S = Ra \setminus R$ as in the proof of theorem 2.16. Since $R \subset Ra \subset F$, then $Q \subset F$. Therefore, $R_1 = Q \cap F = Q = Ra \setminus R$. Then, the scaling function is supported on R and the wavelet on R_1 .

With $\psi = |\det(c)|^{\frac{1}{2}} \check{\chi}_{R_1}$, we have

$$\{D_a^j D_b T_k \psi : j \in \mathbb{Z}, b \in B, k \in \Gamma\}$$

is a singly generated, MSF, MRA, composite dilation Parseval frame wavelet for $L^2(\mathbb{R}^n)$. \square

Figure 2.19 depicts a singly generated, MSF, MRA, composite dilation Parseval frame wavelet. Let $B = \langle r_1, r_2 \rangle$ be the group generated by reflections through $y = x$ and $y = 0$. This is the group of symmetries of the square. Let $\Gamma = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mathbb{Z}^2$ and $a = 2(I_2)$. Then taking P to be the parrallogram formed by the two basis vectors for Γ^* , we construct R by $R = Pa^{-1}$ as in the proof above. We let $R_1 = Ra \setminus R$. Then the scaling function, φ , is defined by $\hat{\varphi} = \chi_R$ and the wavelet, ψ , is defined by

$$\hat{\psi} = \chi_{R_1}.$$

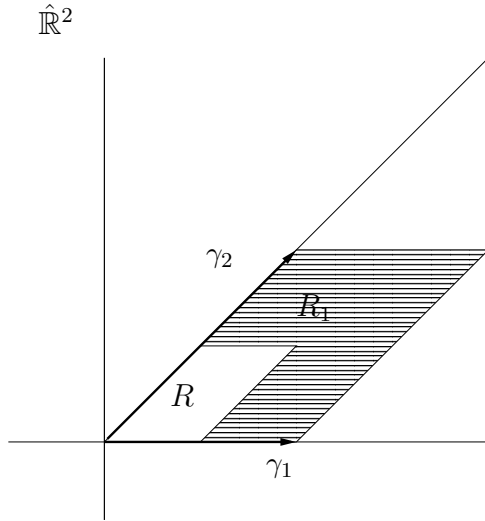


Figure 2.19: A singly generated, MSF, MRA, composite dilation Parseval frame wavelet.

This shows us a particular way to construct such composite dilation Parseval frame wavelets. Since we seek only subsets of the lattice tiling sets, this also significantly loosens the restriction on the geometry of the supporting sets. In the orthonormal case, we always ended with some union of parallelepipeds. However, here we may take any geometry we desire as long as the B -tiling of the set supporting the scaling function is a starlike neighborhood of the origin. For example, we may choose a ball.

Example 2.3. Let B be a group generated by reflections through n hyperplanes through the origin. Let $\Gamma = c\mathbb{Z}^n$ be a lattice such that the basis elements of Γ^* each lie in a unique bounding hyperplane. Let $t = \min \{|\gamma_i| : i = 1, \dots, n\}$. Define S to be the n dimensional ball centered at the origin with radius $\frac{1}{2}t$. A ball centered at the origin is invariant under B . Let $R = S \cap F$ where F is a fundamental domain for B . Then with R and S , Γ is PF- B -admissible since $R \subset \{\sum_{i=1}^n t_i \gamma_i : t_i \in [0, 1]\}$ and $S = \bigcup_{b \in B} Rb$ is obviously a starlike neighborhood of the origin.

Let $a = 2I_n$. Then $R \subset Ra$ and since Ra has radius t , $t \leq |\gamma_i|$ for all $i = 1, \dots, n$,

$Ra \subset \{\sum_{i=1}^n t_i \gamma_i : t_i \in [0, 1]\}$. Now, with $R_1 = Ra \setminus R$,

$$Sa = \bigcup_{b \in B} Rba = \bigcup_{b \in B} Rab = \left\{ \bigcup_{b \in B} R_1 b \right\} \cup \left\{ \bigcup_{b \in B} Rb \right\} = \left\{ \bigcup_{b \in B} R_1 b \right\} \cup S.$$

Thus $S \subset Sa$, R_1 is contained in a Γ^* tiling set for $\hat{\mathbb{R}}^n$, and R_1 is a B -tiling set for $Sa \setminus S$. Therefore, a is PF- (B, Γ) -admissible. Hence, with $\psi = |\det(c)|^{\frac{1}{2}} \check{\chi}_{R_1}$, we have

$$\{D_a^j D_b T_k \psi : j \in \mathbb{Z}, b \in B, k \in \Gamma\}$$

is an MSF, MRA, composite dilation Parseval frame wavelet for $L^2(\mathbb{R}^n)$.

Figure 2.20 depicts such a wavelet in \mathbb{R}^2 when B is the group of symmetries of the square, $\Gamma = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mathbb{Z}^2$ and $a = 2(I_2)$. In this case the scaling function, φ , is defined by $\hat{\varphi} = \chi_R$ and the wavelet, ψ , is defined by $\hat{\psi} = \chi_{R_1}$.

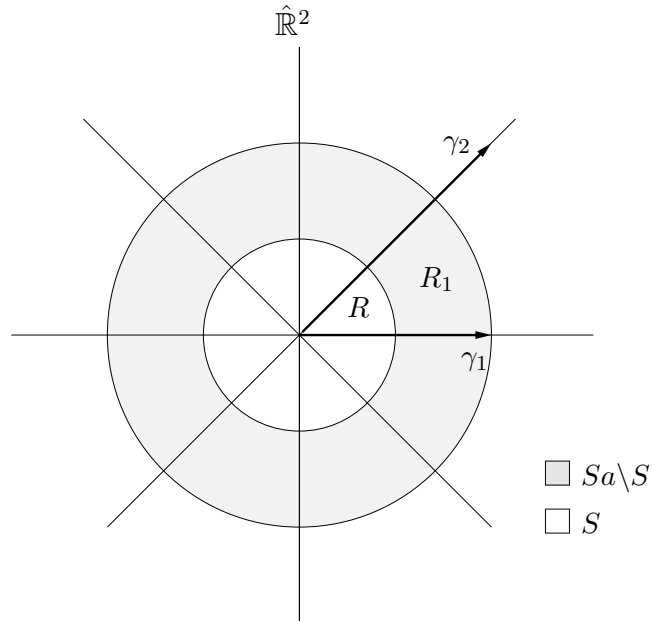


Figure 2.20: A singly generated, MSF, MRA, composite dilation Parseval frame wavelet from a sphere.

Next we present an example due to Alexi Savov [14] which provides a singly

generated, MRA, MSF composite dilation Parseval frame wavelet for $L^2(\mathbb{R}^3)$. This system exploits the symmetries of a regular polygon in \mathbb{R}^2 by looking at a double pyramid. This gives us an example in three dimensions that has a potentially useful geometric structure.

Example 2.4. Let P be a regular n -gon in $\hat{\mathbb{R}}^2$, $n > 4$. We will denote $\hat{\mathbb{R}}^2$ as the ξ_1, ξ_2 plane. Let \tilde{B} be the group of symmetries of P and \tilde{P} a \tilde{B} -tiling set for P under the action of B , $P = \bigcup_{b \in B} \tilde{P}b$ is a disjoint union.

Figure 2.21 shows \tilde{P} and P when $n = 5$. When B is the group of symmetries of a pentagon, the triangle \tilde{P} is a \tilde{B} -tiling set for the darker pentagon P .

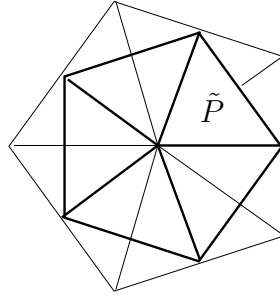


Figure 2.21: For $n = 5$, \tilde{P} is a \tilde{B} -tiling set for P and $P \subset Pa$.

Select a point $h = (0, 0, h)$ on the ξ_3 axis and generate the pyramid, Q , determined by P and h . Define R as the convex hull of \tilde{P} and h . Then $Q = \bigcup_{\tilde{b} \in \tilde{B}} R\tilde{b}$.

Let B be the group obtained by adding the reflection through the ξ_1, ξ_2 plane to \tilde{B} ; with r the reflection through the ξ_1, ξ_2 plane,

$$r = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

we may write $B = \langle \tilde{B}, r \rangle$. Also, we let $S = Q \cup Qr$. Then B fixes S and $S = \bigcup_{b \in B} Rb$.

Let

$$a = \begin{bmatrix} 1 & \tan(\frac{\pi}{n}) & 0 \\ -\tan(\frac{\pi}{n}) & 1 & 0 \\ 0 & 0 & 2 \cos^2(\frac{\pi}{n}) \end{bmatrix}. \quad (2.73)$$

From (2.73), we see that we can write a as the product of a rotation by $-\frac{\pi}{n}$ in the ξ_1, ξ_2 plane and an expanding matrix. Namely,

$$a = \rho\alpha = \begin{bmatrix} \cos(\frac{\pi}{n}) & \sin(\frac{\pi}{n}) & 0 \\ -\sin(\frac{\pi}{n}) & \cos(\frac{\pi}{n}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\cos(\frac{\pi}{n})} & 0 & 0 \\ 0 & \frac{1}{\cos(\frac{\pi}{n})} & 0 \\ 0 & 0 & 2 \cos^2(\frac{\pi}{n}) \end{bmatrix}. \quad (2.74)$$

Since we chose $n > 4$, the matrix α is expanding. With $n > 4$, $|\cos(\frac{\pi}{n})| > \frac{1}{\sqrt{2}}$. Therefore, the eigenvalues of α are $|\frac{1}{\cos(\frac{\pi}{n})}| > \sqrt{2}$ and $|2 \cos^2(\frac{\pi}{n})| > 1$. Since ρ is a rotation, $|\det(\rho)| = 1$. So a is the composition of a rotation in the ξ_1, ξ_2 plane and an expanding matrix, and is therefore expanding.

Now we observe that a normalizes B . We write $B = aBa^{-1} = \rho\alpha B\alpha^{-1}\rho^{-1}$. Since α is diagonal, α will commute with every element of the group B . Thus $B = \rho B\rho^{-1}$. However, ρ is a rotation by $-\frac{\pi}{n}$ in the ξ_1, ξ_2 plane and, therefore, as discussed in section 2.4.3, ρ normalizes \tilde{B} . Since r fixes $\hat{\mathbb{R}}^2$, ρ and r commute. Thus ρ normalizes $B = \langle \tilde{B}, r \rangle$.

Finally, we show that $S \subset Sa$. First we observe the action of a^{-1} on P , the n -gon base of S in the ξ_1, ξ_2 plane. Let, $v = (v_1, v_2, 0), w = (w_1, w_2, 0)$ be any two adjacent vertices of P . Then $|v| = |w|$. Also $\angle v w = \frac{2\pi}{n}$. Hence, the midpoint of the edge of P determined by v and w is defined by a vector of length $\cos(\frac{\pi}{n})|v|$. Then the inscribed circle of P has radius $\cos(\frac{\pi}{n})|v|$.

For all $\xi_0 = (\xi_1, \xi_2, 0) \in P$, we know $|\xi| \leq |v|$. Since $a^{-1} = \alpha^{-1}\rho^{-1}$ and ρ^{-1} is a rotation of P , then $|\xi_0 a^{-1}| = |\xi_0 \alpha^{-1}| = \cos(\frac{\pi}{n})|\xi_0| < \cos(\frac{\pi}{n})|v|$. Thus, Pa^{-1} is contained in the inscribed circle of P establishing $Pa^{-1} \subset P$.

Figure 2.21 depicts $P \subset Pa$ for a pentagon base. With a pentagon base, $n = 5$ and a is defined by equation (2.74).

Now we observe the action of a^{-1} on S . Recall that Q was the pyramid determined by P and a point $(0, 0, h)$. We observe the

$$|(0, 0, h)a^{-1}| = |(0, 0, h)\alpha^{-1}| = \frac{1}{2 \cos^2(\frac{\pi}{n})} |h| < |h|.$$

Since a^{-1} is also the product of a rotation in the ξ_1, ξ_2 plane and a diagonal matrix, then a^{-1} will take the convex hull of P and $(0, 0, h)$ to the convex hull of Pa^{-1} and $(0, 0, h)a^{-1}$. With $Pa^{-1} \subset P$ and $|(0, 0, h)a^{-1}| < |h|$, then $Qa^{-1} \subset Q$. Now

$$Sa^{-1} = [Q \cup Qr] a^{-1} = \{Qa^{-1}\} \cup \{Qra^{-1}\}.$$

Since r is diagonal, $ra^{-1} = a^{-1}r$. So we have

$$Sa^{-1} = \{Qa^{-1}\} \cup \{Qa^{-1}r\} = \{Qa^{-1}\} \cup \{Qa^{-1}\}r \subset Q \cup Qr = S.$$

Therefore, $S \subset Sa$.

With our example for $n = 5$, figure 2.22 shows the relation $S \subset Sa$ for a defined by equation (2.74).

Now we have a set R that is a B -tiling set of S . We need to choose a lattice $\Gamma = c\mathbb{Z}^n$ such that R is contained a Γ^* -tiling set of $\hat{\mathbb{R}}^n$. With such a lattice, Γ is PF- B -admissible.

Since a is expanding, $S \subset Sa$, and a normalizes B , a is PF- (B, Γ) -admissible. However, to obtain a singly generated composite dilation wavelet, we also need that $R_1 = Ra \setminus S$ is contained in a lattice tiling set of $\hat{\mathbb{R}}^n$. Therefore, we simply choose Γ so that both R and R_1 are contained in lattice tiling sets.

Alternatively, we can simply scale S such that R and R_1 are contained in $[0, 1]^3$.

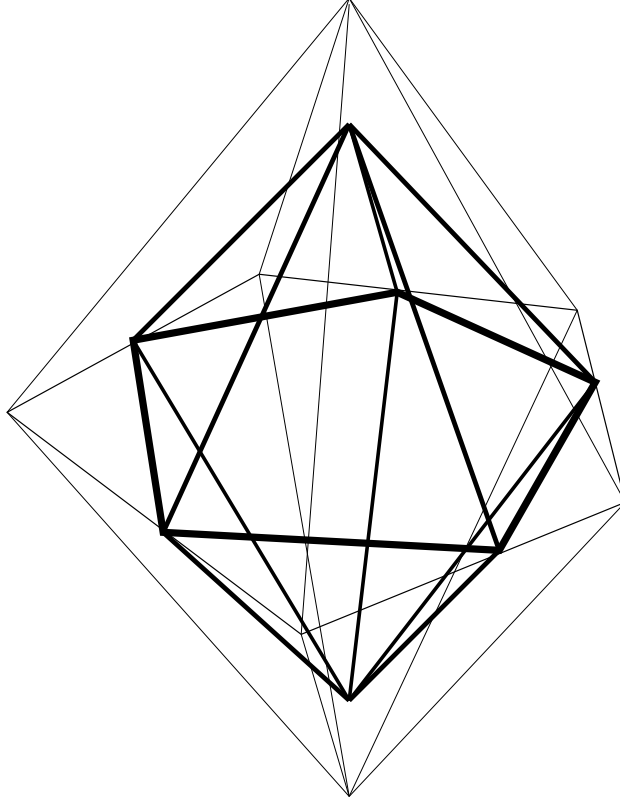


Figure 2.22: $S \subset Sa$ for a double pyramid with pentagon base.

Choose P such that the length of any vertex is $\cos(\frac{\pi}{n})$. Then Pa has vertices of length 1. Additionally, choose $h = \cos^{-2}(\frac{\pi}{n})$. With $n > 4$, we can choose $R, R_1 \subset [0, 1]^3$. If we choose P so that one of the vertices lies on the positive ξ_1 axis, then the largest S such that $R, R_1 \subset [0, 1]^3$ is the double pyramid with vertices

$$\left\{ \cos\left(\frac{\pi}{n}\right) \cos\left(\frac{2\pi}{n}j\right) : j = 0, \dots, n-1 \right\} \cup \left\{ (0, 0, \pm \frac{1}{\cos^2(\frac{\pi}{n})}) \right\}.$$

This section highlights the ease with which we may obtain MRA, MSF, composite dilation Parseval frame wavelets. Because the measure of the support sets for the scaling function and wavelet are no longer important, it is always the case that we may take a Parseval frame composite dilation wavelet and reduce the number of

generators to one. To do so, we may simply scale the original support of the scaling function so that it and the support of the wavelet are contained in a lattice tiling. With this in mind, relaxing the requirement of an orthonormal basis to that of a Parseval frame may be extremely useful.

Chapter 3

Accuracy of Compactly Supported Composite Dilation Wavelets

From Cabrelli, Heil, and Molter, [2] we have necessary conditions for the existence of a system generated by several, multidimensional functions to have a specified level of accuracy. Here we are concerned with the situation where the several functions are defined by a group acting on a single function by dilation. That is, if φ is an integrable function and $B \in GL_d(\mathbb{R})$ is a group such that $|B| = r < \infty$, then $\varphi_b(x) = D_b\varphi(x) = \varphi(b^{-1}x)$. When we fix a specific ordering to the list of elements in B , namely $B = \{I, b_2, \dots, b_r\}$ we can write $\varphi_i = \varphi_{b_i} = D_{b_i}\varphi$ where $\varphi_1 = D_I\varphi = \varphi$. Then, writing the r functions as a column vector, we have $\Phi(x) = (\varphi_1(x), \varphi_2(x), \dots, \varphi_r(x))^t$.

We want $\Phi(x)$ to have compact support and be a refinable function, i.e.

$$\Phi(x) = \sum_{k \in \Lambda} c^k \Phi(Ax - k) \quad (3.1)$$

for some $\Lambda \subset \Gamma$, Λ finite, some $r \times r$ matrices c^k , and A an expanding matrix. We also require Λ to be invariant under the action of B , namely $B(\Lambda) = \Lambda$. In this case, we call $\{\Phi_{j,i,k}\} = \{D_A^j D_{b_i} T_k \varphi \mid i = 1, \dots, r; j, k \in \mathbb{Z}\}$ a compactly supported, refinable *Composite Dilation System*.

First of all, we want to show that $\Phi : \mathbb{R}^d \rightarrow \mathbb{C}^r$ is integrable. Since $\varphi \in L^1(\mathbb{R}^d)$, we have $\|\varphi\|_1 < \infty$. With $B \subset GL_d(\mathbb{R})$ and $|B| = r < \infty$ then for all $b \in B$ we know $|\det(b)| = 1$. Thus, for all $b \in B$,

$$\begin{aligned} \|\varphi_b\|_1 &= \int_{\mathbb{R}^d} |\varphi_b(x)| dx = \int_{\mathbb{R}^d} |\varphi(b^{-1}x)| dx = \int_{b\mathbb{R}^d} |\varphi(y)| |\det(b)| dy \\ &= \int_{\mathbb{R}^d} |\varphi(y)| dy = \|\varphi\|_1 < \infty. \end{aligned}$$

Therefore, $\varphi_b \in L^1(\mathbb{R}^d)$ for all $b \in B$. Hence, $\Phi \in L^1(\mathbb{R}^d, \mathbb{C}^r)$.

3.1 Matrix Relations from Composite Dilations

This section introduces a simplification for composite dilation systems to the necessary conditions provided in [2]. As we mentioned at the end of section 1.3, we may take the point of view that composite dilation systems have $r = |B|$ generating functions related by the composite dilation group, B . This relationship from the group action defines a set of identities for the entries of the coefficient matrices from the refinement equation. Therefore, we are able to significantly reduce the number of free entries in our coefficient matrices by establishing matrix relations for composite dilation systems.

Suppose $\Phi(x)$ is a compactly supported refinable function. Then there exist $\Lambda \subset \Gamma$, Λ finite, a set of $r \times r$ matrices c^k (where $c^k = 0$ for $k \notin \Lambda$), and an expanding matrix A such that

$$\Phi(x) = \sum_{k \in \Lambda} c^k \Phi(Ax - k).$$

Then

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \vdots \\ \varphi_r(x) \end{pmatrix} = \sum_{k \in \Lambda} c^k \begin{pmatrix} \varphi_1(Ax - k) \\ \varphi_2(Ax - k) \\ \vdots \\ \varphi_r(Ax - k) \end{pmatrix} = \begin{pmatrix} \sum_{k \in \Lambda} \sum_{j=1}^r c_{1,j}^k \varphi_j(Ax - k) \\ \sum_{k \in \Lambda} \sum_{j=1}^r c_{2,j}^k \varphi_j(Ax - k) \\ \vdots \\ \sum_{k \in \Lambda} \sum_{j=1}^r c_{r,j}^k \varphi_j(Ax - k) \end{pmatrix}.$$

This Composite Dilation System places certain relations on the entries of the $r \times r$ matrices c^k . We know from the above equation that for each $i = 1, \dots, r$ we have

$$\varphi_i(x) = \sum_{k \in \Lambda} \sum_{j=1}^r c_{i,j}^k \varphi_j(Ax - k).$$

Simultaneously, we know that φ_i is defined by the action of b_i providing

$$\varphi_i(x) = \varphi_1(b_i^{-1}x) = \sum_{k \in \Lambda} \sum_{j=1}^r c_{1,j}^k \varphi_j(Ab_i^{-1}x - k).$$

So we have the following equation:

$$\sum_{k \in \Lambda} \sum_{j=1}^r c_{i,j}^k \varphi_j(Ax - k) = \sum_{k \in \Lambda} \sum_{j=1}^r c_{1,j}^k \varphi_j(Ab_i^{-1}x - k).$$

Suppose that A normalizes B . That is, for each $b \in B$ there exists $\tilde{b} \in B$ such that $\tilde{b} = AbA^{-1}$. Then, writing $\varphi_{b_i} = \varphi_i$, we have

$$\begin{aligned} \sum_{k \in \Lambda} \sum_{l=1}^r c_{i,l}^k \varphi_{b_i}(Ax - k) &= \sum_{k \in \Lambda} \sum_{j=1}^r c_{1,j}^k \varphi_{b_j}(Ab_i^{-1}x - k) \\ &= \sum_{k \in \Lambda} \sum_{j=1}^r c_{1,j}^k \varphi_{b_j}(\tilde{b}_i^{-1}Ax - k) \\ &= \sum_{k \in \Lambda} \sum_{j=1}^r c_{1,j}^k \varphi_{b_j}(\tilde{b}_i^{-1}(Ax - \tilde{b}_i k)) \end{aligned}$$

$$\begin{aligned}
&= \sum_{k \in \Lambda} \sum_{j=1}^r c_{1,j}^k \varphi_{\tilde{b}_i b_j}(Ax - \tilde{b}_i k) \\
&= \sum_{k \in \Lambda} \sum_{j=1}^r c_{1,j}^{\tilde{b}_i^{-1} k} \varphi_{\tilde{b}_i b_j}(Ax - k).
\end{aligned}$$

Since B is a group, there exists $b_l \in B$ such that $\tilde{b}_i b_j = b_l$. In each case we have

$$c_{i,l}^k = c_{1,j}^{\tilde{b}_i^{-1} k} \quad \text{or} \quad c_{1,j}^k = c_{i,l}^{\tilde{b}_i k}. \quad (3.2)$$

Therefore, the free entries in the $r \times r$ matrices c^k are only the elements of the first row of the matrix. So when attempting to find appropriate matrices for the refinement equation, we have $|B||\Lambda| = r|\Lambda|$ free variables.

Example 3.1. Let $d = 1$, $B = \{b_1, b_2\} = \{1, -1\}$, $\Lambda = \{-n, \dots, 0, \dots, n\} \subset \mathbb{Z}$, $A = 2$. Here $\Phi(x) = (\varphi(x), D_{-1}\varphi(x))^t = (\varphi(x), \varphi(-x))^t$. We see that $|B| = 2$ and $|\Lambda| = 2n + 1$ so there are $4n + 2$ free entries for the associated matrices. From (3.2), we have the general relations $c_{1,j}^k = c_{i,l}^{\tilde{b}_i k}$. Here B consists of the two scalars 1 and -1 and A is the scalar 2 so A and B commute. Thus $\tilde{b} = b$ for all $b \in B$. Then:

$$\begin{aligned}
\tilde{b}_1 b_1 &= b_1 b_1 = 1 \cdot 1 = 1 = b_1 \\
\tilde{b}_1 b_2 &= b_1 b_2 = 1 \cdot -1 = -1 = b_2 \\
\tilde{b}_2 b_1 &= b_2 b_1 = -1 \cdot 1 = -1 = b_2 \\
\tilde{b}_2 b_2 &= b_2 b_2 = -1 \cdot -1 = 1 = b_1
\end{aligned}$$

Hence, for $j = 1$ we have $l = i$, and for $j = 2$ we have $l = (i \bmod 2) + 1$. So

$$c_{1,1}^k = c_{1,1}^k, \quad c_{1,1}^k = c_{2,2}^{-k}, \quad c_{1,2}^k = c_{1,2}^k, \quad c_{1,2}^k = c_{2,1}^{-k}.$$

Now dropping the comma in the subscript, the useful relations are

$$c_{11}^k = c_{22}^{-k}, \quad c_{12}^k = c_{21}^{-k}. \quad (3.3)$$

Therefore, for $|k| < n$, the matrices for this system are:

$$c^{-k} = \begin{pmatrix} c_{11}^{-k} & c_{12}^{-k} \\ c_{12}^k & c_{11}^k \end{pmatrix}, \quad c^0 = \begin{pmatrix} c_{11}^0 & c_{12}^0 \\ c_{12}^0 & c_{11}^0 \end{pmatrix}, \quad c^k = \begin{pmatrix} c_{11}^k & c_{12}^k \\ c_{12}^{-k} & c_{11}^{-k} \end{pmatrix},$$

Here we take the free variables to be the entries of the first row of the matrix associated to each lattice point in Λ . Alternatively, since $c_{21}^k = c_{12}^{-k}$ and $c_{22}^k = c_{11}^{-k}$, we could look at the free variables as the entries of the matrices associated with the nonnegative lattice points in Λ . Suppose $n = 2$, so $\Lambda = \{-2, -1, 0, 1, 2\}$; then these matrices would be

$$c^0 = \begin{pmatrix} c_{11}^0 & c_{12}^0 \\ c_{12}^0 & c_{11}^0 \end{pmatrix}, \quad c^1 = \begin{pmatrix} c_{11}^1 & c_{12}^1 \\ c_{21}^1 & c_{22}^1 \end{pmatrix}, \quad c^2 = \begin{pmatrix} c_{11}^2 & c_{12}^2 \\ c_{21}^2 & c_{22}^2 \end{pmatrix}.$$

Then the remaining matrices, c^{-1} and c^{-2} , are defined by the relations, namely

$$c^{-1} = \begin{pmatrix} c_{22}^1 & c_{21}^1 \\ c_{12}^1 & c_{11}^1 \end{pmatrix}, \quad c^{-2} = \begin{pmatrix} c_{22}^2 & c_{21}^2 \\ c_{12}^2 & c_{11}^2 \end{pmatrix}.$$

Example 3.2. Let $d = 2$, $B = \{b_1, b_2, b_3, b_4\} = \{I, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, -I\}$,

and $A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. Here $\Phi(x) = (\phi(x), D_{b_2}\phi(x), D_{b_3}\phi(x), D_{b_4}\phi(x))^t$. We see that $|B| = 4$. We first observe that $\tilde{b}_1 = b_1$, $\tilde{b}_2 = b_3$, $\tilde{b}_3 = b_2$, and $\tilde{b}_4 = b_4$ where

$\tilde{b}_j = Ab_jA^{-1}$. From (3.2), we obtain the following identities for B :

$$\begin{array}{cccc}
\tilde{b}_1b_1 = b_1 & \tilde{b}_1b_2 = b_2 & \tilde{b}_1b_3 = b_3 & \tilde{b}_1b_4 = b_4 \\
\tilde{b}_2b_1 = b_3 & \tilde{b}_2b_2 = b_4 & \tilde{b}_2b_3 = b_1 & \tilde{b}_2b_4 = b_2 \\
\tilde{b}_3b_1 = b_2 & \tilde{b}_3b_2 = b_1 & \tilde{b}_3b_3 = b_4 & \tilde{b}_3b_4 = b_3 \\
\tilde{b}_4b_1 = b_4 & \tilde{b}_4b_2 = b_3 & \tilde{b}_4b_3 = b_2 & \tilde{b}_4b_4 = b_1.
\end{array}$$

Therefore, using (3.2) we find the coefficient identities. For example, when $i = 2$ and $j = 1$, we have $c_{11}^k = c_{23}^k = c_{23}^{b_2k} = c_{23}^{b_3k}$. We have the following identities.

$$\begin{aligned}
c_{11}^k &= c_{23}^{b_3k} = c_{32}^{b_2k} = c_{44}^{b_4k} \\
c_{12}^k &= c_{24}^{b_3k} = c_{31}^{b_2k} = c_{43}^{b_4k} \\
c_{13}^k &= c_{21}^{b_3k} = c_{34}^{b_2k} = c_{42}^{b_4k} \\
c_{14}^k &= c_{22}^{b_3k} = c_{33}^{b_2k} = c_{41}^{b_4k}
\end{aligned}$$

This leads us to an explicit definition of the coefficient matrices, namely,

$$c^k = \begin{pmatrix} c_{11}^k & c_{12}^k & c_{13}^k & c_{14}^k \\ c_{13}^{b_3k} & c_{14}^{b_3k} & c_{11}^{b_3k} & c_{12}^{b_3k} \\ c_{11}^{b_2k} & c_{12}^{b_2k} & c_{13}^{b_2k} & c_{14}^{b_2k} \\ c_{14}^{b_4k} & c_{13}^{b_4k} & c_{12}^{b_4k} & c_{11}^{b_4k} \end{pmatrix}. \quad (3.4)$$

Now let us pick a specific subset of the lattice \mathbb{Z}^2 . Let

$$\Lambda = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}.$$

Recall that in the time domain, the points in \mathbb{R}^2 are written as column vectors. We see that $B(\Lambda) = \Lambda$. Also, for $\tilde{\Lambda} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, we have $B(\tilde{\Lambda}) = \Lambda$. We know we

have $|B| |\Lambda| = 4 \cdot 5 = 20$ free entries for the 4×4 coefficient matrices.

To simplify the notation, we use α for the entries of $c^{(1)}$, β for the entries of $c^{(0)}$, and δ for the entries of $c^{(0)}$. Since $\binom{1}{0} = b_3 \binom{1}{0}$, the entries in the first and third rows of $c^{(1)}$ are free variables. Therefore, 8 of our choices are the matrix entries α_{1j} and α_{3j} for $j = 1, 2, 3, 4$. Since $b_2 \binom{0}{1} = b_4 \binom{0}{1}$, the entries in the first and second rows of $c^{(0)}$ are free variables. Therefore, an additional 8 choices are the matrix entries β_{1j} and β_{2j} for $j = 1, 2, 3, 4$. Since $c^{(0)}$ is completely determined by the entries of its first row, the remaining 4 free variables for our matrix entries are δ_{1j} for $j = 1, 2, 3, 4$. Then our twenty free entries are $\{\alpha_{1j}, \alpha_{3j}, \beta_{1j}, \beta_{2j}, \delta_{1j} : j = 1, 2, 3, 4\}$. Thus, the matrices for this example are

$$c^{(1)} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{13} & \alpha_{14} & \alpha_{11} & \alpha_{12} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{33} & \alpha_{34} & \alpha_{31} & \alpha_{32} \end{pmatrix} \quad c^{(0)} = \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \\ \beta_{12} & \beta_{11} & \beta_{14} & \beta_{13} \\ \beta_{22} & \beta_{21} & \beta_{24} & \beta_{23} \end{pmatrix} \quad (3.5)$$

$$c^{(0)} = \begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} & \delta_{14} \\ \delta_{13} & \delta_{14} & \delta_{11} & \delta_{12} \\ \delta_{12} & \delta_{11} & \delta_{14} & \delta_{13} \\ \delta_{14} & \delta_{13} & \delta_{12} & \delta_{11} \end{pmatrix}.$$

The two remaining matrices are now defined by the matrices above.

$$c^{(-1)} = \begin{pmatrix} \alpha_{32} & \alpha_{31} & \alpha_{34} & \alpha_{33} \\ \alpha_{34} & \alpha_{33} & \alpha_{32} & \alpha_{31} \\ \alpha_{12} & \alpha_{11} & \alpha_{14} & \alpha_{13} \\ \alpha_{14} & \alpha_{13} & \alpha_{12} & \alpha_{11} \end{pmatrix} \quad c^{(0)} = \begin{pmatrix} \beta_{23} & \beta_{24} & \beta_{21} & \beta_{22} \\ \beta_{13} & \beta_{14} & \beta_{11} & \beta_{12} \\ \beta_{24} & \beta_{23} & \beta_{22} & \beta_{21} \\ \beta_{14} & \beta_{13} & \beta_{12} & \beta_{11} \end{pmatrix}.$$

3.2 Accuracy of Composite Dilation Systems in 1 Dimension

In this section we identify the necessary conditions for a composite dilation system in 1 dimension to have a given level of accuracy. The one dimensional case is a natural place to begin the search for more accurate composite dilation systems. We are not yet looking for orthonormal bases, but simply establishing when an affine system with composite dilations will have arbitrary accuracy. We establish that such a system exists for every level of accuracy.

Let us restrict our attention to Composite Dilation Systems for $L^2(\mathbb{R})$. We need a group B consisting of elements with $|b| = 1$ for all $b \in B$. In 1 dimension, the only such elements are 1 and -1. Thus, $B = \{1, -1\}$. Let $\Gamma = \mathbb{Z}$ since for any $\Gamma = c\mathbb{Z}$ with $c \in \mathbb{R} = GL_1(\mathbb{R})$ we have $\Gamma c^{-1} = \mathbb{Z}$. From 3.1, we have the relations 3.3 $c_{11}^k = c_{22}^{-k}$, $c_{12}^k = c_{21}^{-k}$. From [2] Theorem (?), in order to have a system with accuracy p , we must find row vectors $v_{[s]}$; $0 \leq s < p$ such that

$$v_{[s]} = \sum_{k \in \Gamma_i} \sum_{t=0}^s Q_{[s,t]}(k) A_{[t]} v_{[t]} c^k.$$

In one dimension we have $v_{[s]} = v_s$ and $A_{[s]} = A^s$ for all $0 \leq s < p$. We also know that $Q_{[s,t]}(k) = \sum_{t=0}^s (-1)^{s-t} \binom{s}{t} k^{s-t}$. Therefore, in $L^2(\mathbb{R})$ we are searching for row vectors $\{v_s \in \mathbb{C}^{1 \times r} | 0 \leq s < p\}$ with $v_0 \neq 0$ and

$$v_s = \sum_{k \in \Gamma_i} \sum_{t=0}^s (-k)^{s-t} \binom{s}{t} A^t v_t c^k.$$

Since $B = \{1, -1\}$, then $r = |B| = 2$ and $\Phi(x) = (\varphi(x), \varphi(-x))^t$. When we take the traditional dyadic dilation, $A = 2$, then from [2], we combine theorems 3.4 and 3.6 of Cabrelli, Heil, and Molter. In our case, this can be written as follows:

Theorem (Cabrelli, Heil, Molter, [2]): Assume $\Phi(x) = (\varphi(x), \varphi(-x))^t : \mathbb{R} \rightarrow \mathbb{C}^2$ is a refinable, integrable, compactly supported function with independent \mathbb{Z} -translates. Then Φ has accuracy p if and only if there exists a collection of row vectors $\{v_s \in \mathbb{C}^{1 \times 2} | 0 \leq s < p\}$ such that

$$(i) v_0 \hat{\Phi}(0) \neq 0, \text{ and}$$

$$(ii) v_s = \sum_{k \in \Gamma_i} \sum_{t=0}^s (-k)^{s-t} \binom{s}{t} 2^t v_t c^k.$$

Now for some $\alpha \in \mathbb{R}$, we know $(2\alpha - k)^s = \sum_{t=0}^s \binom{s}{t} (-k)^{s-t} 2^t \alpha^t$. Define, for $0 \leq t < p$,

$$v_t = \alpha^t (1, (-1)^t).$$

Then

$$\begin{aligned} v_s &= \sum_{k \in \Gamma_i} \sum_{t=0}^s \binom{s}{t} (-k)^{s-t} 2^t v_t c^k \\ &= \sum_{k \in \Gamma_i} \sum_{t=0}^s \binom{s}{t} (-k)^{s-t} 2^t \alpha^t (\alpha^{-t} v_t) c^k. \end{aligned}$$

$\alpha^{-t} v_t = (1, (-1)^t)$ so

$$\begin{aligned} (\alpha^{-t} v_t) c^k &= (1, (-1)^t) \begin{pmatrix} c_{11}^k & c_{12}^k \\ c_{21}^k & c_{22}^k \end{pmatrix} \\ &= [c_{11}^k + (-1)^t c_{21}^k, c_{12}^k + (-1)^t c_{22}^k]. \end{aligned}$$

Then

$$\begin{aligned} v_s &= \sum_{k \in \Gamma_i} \sum_{t=0}^s \binom{s}{t} (-k)^{s-t} (2\alpha)^t [c_{11}^k + (-1)^t c_{21}^k, c_{12}^k + (-1)^t c_{22}^k] \\ &= \sum_{k \in \Gamma_i} \left[\sum_{t=0}^s \binom{s}{t} (-k)^{s-t} (2\alpha)^t c_{11}^k + \sum_{t=0}^s \binom{s}{t} (-k)^{s-t} (2\alpha)^t (-1)^t c_{21}^k, \right. \\ &\quad \left. \sum_{t=0}^s \binom{s}{t} (-k)^{s-t} (2\alpha)^t c_{12}^k + \sum_{t=0}^s \binom{s}{t} (-k)^{s-t} (2\alpha)^t (-1)^t c_{22}^k \right] \end{aligned}$$

$$\begin{aligned}
& \left[\sum_{t=0}^s \binom{s}{t} (-k)^{s-t} (2\alpha)^t c_{12}^k + \sum_{t=0}^s \binom{s}{t} (-k)^{s-t} (2\alpha)^t (-1)^t c_{22}^k \right] \\
= & \sum_{k \in \Gamma_i} \left[(2\alpha - k)^s c_{11}^k + \sum_{t=0}^s \binom{s}{t} (k)^{s-t} (-1)^{s-t} (-1)^t (2\alpha)^t c_{21}^k, \right. \\
& \left. (2\alpha - k)^s c_{12}^k + \sum_{t=0}^s \binom{s}{t} (k)^{s-t} (-1)^{s-t} (-1)^t (2\alpha)^t c_{22}^k \right] \\
= & \sum_{k \in \Gamma_i} \left[(2\alpha - k)^s c_{11}^k + (-1)^s (2\alpha + k)^s c_{21}^k, (2\alpha - k)^s c_{12}^k + (-1)^s (2\alpha + k)^s c_{22}^k \right] \\
= & \left[\sum_{k \in \Gamma_i} (2\alpha - k)^s c_{11}^k + \sum_{k \in \Gamma_i} (-1)^s (2\alpha + k)^s c_{21}^k, \right. \\
& \left. \sum_{k \in \Gamma_i} (2\alpha - k)^s c_{12}^k + \sum_{k \in \Gamma_i} (-1)^s (2\alpha + k)^s c_{22}^k \right].
\end{aligned}$$

Now we can use the relations 3.3 of the matrix entries induced by the action of the group B by substituting $c_{11}^k = c_{22}^{-k}$, $c_{12}^k = c_{21}^{-k}$. Since B must fix $\Lambda \subset \mathbb{Z}$, Λ finite, then Λ must have the form $\{-n, \dots, 0, \dots, n\}$. For all $k \notin \Lambda$, we have $c^k = 0$ or $c_{ij}^k = 0$ for every i, j . Therefore, we make the substitutions and to obtain

$$\sum_{k \in \Gamma_i} (-1)^s (2\alpha + k)^s c_{21}^k = \sum_{k \in \Gamma_i} (-1)^s (2\alpha + k)^s c_{12}^{-k} = \sum_{k \in \Gamma_i} (-1)^s (2\alpha - k)^s c_{12}^k.$$

Similarly,

$$\sum_{k \in \Gamma_i} (-1)^s (2\alpha + k)^s c_{22}^k = \sum_{k \in \Gamma_i} (-1)^s (2\alpha - k)^s c_{11}^k.$$

Thus we have the following necessary equation for v_s :

$$\begin{aligned}
v_s = & \left[\sum_{k \in \Gamma_i} (2\alpha - k)^s c_{11}^k + \sum_{k \in \Gamma_i} (-1)^s (2\alpha + k)^s c_{21}^k, \right. \\
& \left. \sum_{k \in \Gamma_i} (2\alpha - k)^s c_{12}^k + \sum_{k \in \Gamma_i} (-1)^s (2\alpha + k)^s c_{22}^k \right] \\
= & \left[\sum_{k \in \Gamma_i} (2\alpha - k)^s c_{11}^k + \sum_{k \in \Gamma_i} (-1)^s (2\alpha - k)^s c_{12}^k, \right. \\
& \left. \sum_{k \in \Gamma_i} (-1)^s (2\alpha + k)^s c_{21}^k + \sum_{k \in \Gamma_i} (-1)^s (2\alpha - k)^s c_{22}^k \right]
\end{aligned}$$

$$\begin{aligned}
& \left. \sum_{k \in \Gamma_i} (2\alpha - k)^s c_{12}^k + \sum_{k \in \Gamma_i} (-1)^s (2\alpha - k)^s c_{11}^k \right] \\
&= \sum_{k \in \Gamma_i} [(2\alpha - k)^s (c_{11}^k + (-1)^s c_{12}^k), (2\alpha - k)^s (c_{12}^k + (-1)^s c_{11}^k)] \\
&= \sum_{k \in \Gamma_i} [(2\alpha - k)^s (c_{11}^k + (-1)^s c_{12}^k), (-1)^s (2\alpha - k)^s (c_{11}^k + (-1)^s c_{12}^k)] \\
&= \sum_{k \in \Gamma_i} [1, (-1)^s] (2\alpha - k)^s (c_{11}^k + (-1)^s c_{12}^k). \tag{3.6}
\end{aligned}$$

Since $v_s = \alpha^s(1, (-1)^s)$, we have

$$\alpha^s(1, (-1)^s) = (1, (-1)^s) \sum_{k \in \Gamma_i} (2\alpha - k)^s (c_{11}^k + (-1)^s c_{12}^k).$$

Therefore, for all $0 \leq s < p$,

$$\alpha^s = \sum_{k \in \Gamma_i} (2\alpha - k)^s (c_{11}^k + (-1)^s c_{12}^k). \tag{3.7}$$

This provides the necessary conditions to find our collection of row vectors. We must find $\alpha \in \mathbb{R}$ and the entries of the first row of the matrices associated with lattice points in Λ such that

$$[\alpha^t]_{t=0}^s = \sum_{k \in \Gamma_i} [(2\alpha - k)^t, (-1)^t (2\alpha - k)^t]_{t=0}^s \begin{pmatrix} c_{11}^k \\ c_{12}^k \end{pmatrix}. \tag{3.8}$$

Recall that Γ_i is a coset of $\Gamma/A(\Gamma)$. In this case, with $A = 2$ and $\Gamma = \mathbb{Z}$, we have $\Gamma_1 =$ the odds and $\Gamma_0 =$ the evens. Since $\Lambda \subset \Gamma$ is finite and of the form $\{-n, \dots, 0, \dots, n\}$, and $c_{11}^k = c_{12}^k = 0$ for all $k \notin \Lambda$, we replace Γ_i with Λ_i in equation 3.8.

Suppose n is odd. Then $\Lambda_1 = \{-n, -n + 2, \dots, n - 2, n\}$. Let $m = \lfloor \frac{n+1}{2} \rfloor$. Then

$|\Lambda_1| = 2m$. We can write (3.8) as a product of the following matrices:

$$\vec{\alpha} = \begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \\ \vdots \\ \alpha^s \end{pmatrix}_{(s+1) \times 1}$$

and

$$P_{odd} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ a_{-n} & a_{-(n-2)} & \cdots & a_n & b_{-n} & \cdots & b_{n-2} & b_n \\ a_{-n}^2 & a_{-(n-2)}^2 & \cdots & a_n^2 & b_{-n}^2 & \cdots & b_{n-2}^2 & b_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{-n}^s & a_{-(n-2)}^s & \cdots & a_n^s & b_{-n}^s & \cdots & b_{n-2}^s & b_n^s \end{bmatrix}_{(s+1) \times 4m}$$

where $a_k = (2\alpha - k)$ and $b_k = (-1)(2\alpha - k)$. Let

$$\vec{c}_{odd} = \begin{pmatrix} c_{11}^{-n} \\ c_{11}^{-n+2} \\ \vdots \\ c_{11}^n \\ \\ c_{12}^{-n} \\ c_{12}^{-n+2} \\ \vdots \\ c_{12}^n \end{pmatrix}_{4m \times 1}.$$

Then we have

$$\vec{\alpha} = P_{odd} \vec{c}_{odd}. \quad (3.9)$$

In this case, we see that the $(s+1) \times 4m$ matrix P_{odd} has the form of a Vandermonde matrix, namely:

$$P_{odd} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \beta_1 & \beta_2 & \cdots & \beta_{4m} \\ \beta_1^2 & \beta_2^2 & \cdots & \beta_{4m}^2 \\ \vdots & \vdots & \vdots & \vdots \\ \beta_1^s & \beta_2^s & \cdots & \beta_{4m}^s \end{bmatrix}$$

where the β_i are defined by the polynomials $(2\alpha - k)$ or $-(2\alpha - k)$. If we can invert the matrix P_{odd} , then we can solve equation 3.9 for \vec{c}_{odd} . Thus if $\alpha \in \mathbb{R}$ is such that $2\alpha - k \neq 0$ for all $k \in \Lambda_1$, and $s+1 = 4m$, then

$$\vec{c}_{odd} = P_{odd}^{-1} \vec{\alpha}. \quad (3.10)$$

In the case of $\Lambda_2 = \Lambda \cap \Gamma_2$, the even elements of Λ , the same analysis applies except the matrix P_{even} has different dimensions. Recall that in the above analysis, n was odd. Then $\Lambda_2 = \{-n+1, -n+3, \dots, 0, \dots, n-3, n-1\}$. Let $l = \lfloor \frac{n}{2} \rfloor$. Then there are $2l$ nonzero even numbers in Λ_2 . Therefore, $|\Lambda| = 2l+1$. Then P_{even} is the $(s+1) \times 2(2l+1)$ matrix:

$$\begin{bmatrix} 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & 1 \\ a_{-(n-1)} & \cdots & a_0 & \cdots & a_{n-1} & b_{-(n-1)} & \cdots & b_0 & \cdots & b_{n-1} \\ a_{-(n-1)}^2 & \cdots & a_0^2 & \cdots & a_{n-1}^2 & b_{-(n-1)}^2 & \cdots & b_0^2 & \cdots & b_{n-1}^2 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{-(n-1)}^s & \cdots & a_0^s & \cdots & a_{n-1}^s & b_{-(n-1)}^s & \cdots & b_0^s & \cdots & b_{n-1}^s \end{bmatrix}_{(s+1) \times (4l+2)}$$

where $a_k = (2\alpha - k)$ and $b_k = (-1)(2\alpha - k)$.

We also have a different column vector, c_{even} consisting of the entries of the first

row of the coefficient matrices corresponding to the even elements of Λ .

$$\vec{c}_{even} = \begin{pmatrix} c_{11}^{-n+1} \\ \vdots \\ c_{11}^0 \\ \vdots \\ c_{11}^{n-1} \\ \\ c_{12}^{-n+1} \\ \vdots \\ c_{12}^0 \\ \vdots \\ c_{12}^{n-1} \end{pmatrix}_{4m \times 1} .$$

In order to solve the equation

$$\vec{\alpha} = P_{even} \vec{c}_{even} \quad (3.11)$$

we must still have $2\alpha - k \neq 0$ for all $k \in \Lambda_2$ and we need P_{even} to be square, or $s + 1 = 4l + 2$. When this is true, we have the solution

$$\vec{c}_{even} = P_{even}^{-1} \vec{\alpha}. \quad (3.12)$$

If n is even, the analysis is the same but the matrices change slightly. In P_{odd} , the entries of the first column change from powers of $(2\alpha + n)$ to powers of $(2\alpha + n - 1)$ since the largest odd number in Λ will be $n - 1$. The remaining entries change in a similar fashion. The dimensions of the matrix P_{odd} will remain the same: $(s + 1) \times 4m$ with $m = \lfloor \frac{n+1}{2} \rfloor$. The entries of P_{even} will change similarly with $(2\alpha - n + 1)$ replaced by $(2\alpha - n)$. Again, the dimensions of the matrix P_{even} will remain $(s + 1) \times (4l + 2)$ with $l = \lfloor \frac{n}{2} \rfloor$.

In order to find entries for the 2×2 coefficient matrices c^k that will satisfy the necessary conditions for Φ to have accuracy p , we must find any n such that

$$\min \left\{ 4 \left\lfloor \frac{n+1}{2} \right\rfloor, 4 \left\lfloor \frac{n}{2} \right\rfloor + 2 \right\} \geq p.$$

Then setting $\Lambda = \{-n, \dots, 0, \dots, n\}$ and choosing $\alpha \in \mathbb{R}$ such that $2\alpha - k \neq 0$ for all $k \in \Lambda$, we can solve 3.9 $\vec{\alpha} = P_{\text{odd}} \vec{c}_{\text{odd}}$ and 3.11 $\vec{\alpha} = P_{\text{even}} \vec{c}_{\text{even}}$. Then, for all $k \in \Lambda$, we use these solutions to find the coefficient matrices for the refinement equation:

$$c^k = \begin{pmatrix} c_{11}^k & c_{12}^k \\ c_{12}^{-k} & c_{11}^{-k} \end{pmatrix}.$$

If Φ is a solution to the refinement equation with these coefficient matrices, it will have accuracy p , where p is the largest integer such that

$$p \leq \min \left\{ 4 \left\lfloor \frac{n+1}{2} \right\rfloor, 4 \left\lfloor \frac{n}{2} \right\rfloor + 2 \right\}. \quad (3.13)$$

In the preceding discussion we have proved the following theorem:

Theorem 3.1. *For every $p \in \mathbb{N}$ there exists $\alpha \in \mathbb{R}$, a collection of row vectors $\{v_s \in \mathbb{R}^{1 \times 2} \mid 0 \leq s < p\}$, a finite subset $\Lambda \in \mathbb{Z}$, and a collection of 2×2 matrices c^k such that*

$$(i) \ v_0 = (1, 1) \neq 0 \text{ and}$$

$$(ii) \ v_s = \sum_{k \in \Lambda_i} (2\alpha - k)^s c^k = \sum_{k \in \Lambda_i} \sum_{t=0}^s \binom{s}{t} (-k)^{s-t} 2^t v_t c^k.$$

This theorem states that the necessary conditions from [2] are satisfied for the existence of a refinable composite dilation system of arbitrary accuracy on $L^2(\mathbb{R})$. The problem now becomes finding a sufficient condition for a solution to the refinement equation

$$\Phi(x) = \sum_{k \in \Lambda} c^k \Phi(2x - k).$$

The question of determining such sufficient conditions remains open.

3.3 A Compactly Supported, Composite Dilation Wavelet with Accuracy 2

In the previous section we showed that, for dimension one, we can satisfy the necessary conditions for the existence of a Composite Dilation System with arbitrary accuracy. However, that discussion made no attempt to minimize the support for such a Composite Dilation generating function. Also, it does not give us a means to find a solution to the refinement equation. Additionally, we are most concerned with finding a Composite Dilation Wavelet; a function $\psi \in L^2(\mathbb{R})$ such that $\{D_a^j D_b T_k \psi \mid b \in B, |A| > 1, j, k \in \mathbb{Z}\} = \{\psi(b^{-1}A^j x - k)\}$ is an orthonormal basis for $L^2(\mathbb{R})$. A solution when B is the trivial group is the well-known result of Daubechies compactly supported wavelets [5]. In our case, the situation of composite dilation wavelets, we take $B = \{1, -1\}$.

Using the relations (3.3) on the entries of the coefficient matrices, c_{ij}^k , induced by the action of B from example 3.1 and the results of Cabrelli, Heil, and Molter [2] we arrived at Theorem 3.1. From this we can calculate the collection $\{v_s \in \mathbb{R}^{1 \times 2} \mid 0 \leq s < p\}$ and the 2×2 matrices c^k for any level of accuracy. As expected, as the accuracy of the system increases, so must the size of the support of the generating function. That is, Λ must grow proportionately with p .

In this section we derive the necessary accuracy equations for a 1-dimensional composite dilation system with $\Lambda = \{-n, \dots, 0, \dots, n\}$. To obtain a reproducing system, we must at least satisfy the Smith-Barnwell equation. We develop the necessary and sufficient equations introduced by the Smith-Barnwell equation in terms of the entries of the coefficient matrices. We then conclude the section by directly calculating the matrices c^k and solving the refinement equation (3.1) for $\Lambda = \{-2, -1, 0, 1, 2\}$.

Seeking a composite dilation wavelet, we add the constraints induced by satisfying the Smith-Barnwell induced equations and then solve some orthogonality and normalization equations.

3.3.1 The Accuracy Equations

First, we let $A = 2$, $B = \{1, -1\}$, $\Gamma = \mathbb{Z}$, and $\Phi(x) = (\phi(x), \phi(-x))^t$. We start by taking a general $\Lambda = \{-n, \dots, 0, \dots, n\}$ and setting up the necessary accuracy conditions for a collection $\{v_0, v_1\}$ with $v_0 \neq 0$ and the matrices $c^k = \begin{pmatrix} c_{11}^k & c_{12}^k \\ c_{21}^k & c_{22}^k \end{pmatrix}$. Recall, from Example 1, $c_{11}^k = c_{22}^{-k}$ and $c_{12}^k = c_{21}^{-k}$. Since $A = 2$ and $\Gamma = \mathbb{Z}$ we have $\Gamma_0 = 2\mathbb{Z}$, the evens, and $\Gamma_1 = 2\mathbb{Z} + 1$, the odds. Letting $v_0 = (1, 1)$ we get

$$(1, 1) = (1, 1) \sum_{k \text{ even}} c^k = (1, 1) \sum_{k \text{ odd}} c^k.$$

With the notation

$$c^E = \sum_{k \text{ even}} c^k = \begin{pmatrix} \sum_{k \text{ even}} c_{11}^k & \sum_{k \text{ even}} c_{12}^k \\ \sum_{k \text{ even}} c_{21}^k & \sum_{k \text{ even}} c_{22}^k \end{pmatrix} = \begin{pmatrix} c_{11}^E & c_{12}^E \\ c_{21}^E & c_{22}^E \end{pmatrix}$$

we see that

$$c_{11}^E = \sum_{k \text{ even}} c_{11}^k = \sum_{k \text{ even}} c_{11}^{-k} = \sum_{k \text{ even}} c_{22}^k = c_{22}^E.$$

Similarly we have $c_{12}^E = c_{21}^E$. Hence

$$c^E = \sum_{k \text{ even}} c^k = \begin{pmatrix} c_{11}^E & c_{12}^E \\ c_{12}^E & c_{11}^E \end{pmatrix}.$$

Likewise, for the odds, we arrive at

$$c^O = \sum_{k \text{ odd}} c^k = \begin{pmatrix} c_{11}^O & c_{12}^O \\ c_{12}^O & c_{11}^O \end{pmatrix}.$$

Hence, we have the following two equations:

$$(1, 1) = (1, 1) \sum_{k \text{ even}} c^k = (1, 1) \begin{pmatrix} c_{11}^E & c_{12}^E \\ c_{12}^E & c_{11}^E \end{pmatrix} = (1, 1)(c_{11}^E + c_{12}^E)$$

$$(1, 1) = (1, 1) \sum_{k \text{ odd}} c^k = (1, 1) \begin{pmatrix} c_{11}^O & c_{12}^O \\ c_{12}^O & c_{11}^O \end{pmatrix} = (1, 1)(c_{11}^O + c_{12}^O).$$

Therefore, our first set of accuracy equations for $s = 0$ is

$$c_{11}^E + c_{12}^E = 1 = c_{11}^O + c_{12}^O$$

or

$$\sum_{k \text{ even}} (c_{11}^k + c_{12}^k) = 1 = \sum_{k \text{ odd}} (c_{11}^k + c_{12}^k).$$

Now, for $s = 1$, we know

$$v_1 = \sum_{k \in \Gamma_i} \sum_{t=0}^1 \binom{s}{t} (-k)^{s-t} 2^t v_t c^k = \sum_{k \in \Lambda_i} -k v_0 c^k + \sum_{k \in \Lambda_i} 2 v_1 c^k.$$

Then

$$v_1 \left(2 \left(\sum_{k \in \Lambda_i} c^k \right) - I \right) = v_0 \left(\sum_{k \in \Lambda_i} k c^k \right).$$

With our even and odd notation and $v_0 = (1, 1)$, we have

$$v_1 (2c^E - I) = (1, 1) \left(\sum_{k \text{ even}} k c^k \right) \tag{3.14}$$

and

$$v_1(2c^O - I) = (1, 1) \left(\sum_{k \text{ odd}} kc^k \right). \quad (3.15)$$

Looking at the right hand side of equation (3.14) for k even, we have

$$\begin{aligned} (1, 1) \left(\sum_{k \text{ even}} kc^k \right) &= (1, 1) \left(\sum_{k \in 2\mathbb{N} \cap \Lambda} kc^k - kc^{-k} \right) \\ &= (1, 1) \left[\begin{array}{cc} \sum_{k \in 2\mathbb{N} \cap \Lambda} k(c_{11}^k - c_{11}^{-k}) & \sum_{k \in 2\mathbb{N} \cap \Lambda} k(c_{12}^k - c_{12}^{-k}) \\ \sum_{k \in 2\mathbb{N} \cap \Lambda} k(c_{21}^k - c_{21}^{-k}) & \sum_{k \in 2\mathbb{N} \cap \Lambda} k(c_{22}^k - c_{22}^{-k}) \end{array} \right] \\ &= (1, 1) \left[\begin{array}{cc} \sum_{k \in 2\mathbb{N} \cap \Lambda} k(c_{11}^k - c_{22}^k) & \sum_{k \in 2\mathbb{N} \cap \Lambda} k(c_{12}^k - c_{21}^k) \\ \sum_{k \in 2\mathbb{N} \cap \Lambda} k(c_{21}^k - c_{12}^k) & \sum_{k \in 2\mathbb{N} \cap \Lambda} k(c_{22}^k - c_{11}^k) \end{array} \right] \\ &= \left[\sum_{k \in 2\mathbb{N} \cap \Lambda} k(c_{11}^k - c_{22}^k + c_{21}^k - c_{12}^k), \sum_{k \in 2\mathbb{N} \cap \Lambda} k(c_{12}^k - c_{21}^k + c_{22}^k - c_{11}^k) \right] \\ &= (1, -1) \left(\sum_{k \in 2\mathbb{N} \cap \Lambda} k(c_{11}^k - c_{22}^k + c_{21}^k - c_{12}^k) \right). \end{aligned}$$

Now to simplify the left hand side of equation (3.14) for k even, we use the relation mandated by the accuracy equation for $s = 0$. That is, $c_{11}^E = 1 - c_{12}^E$. Taking $v_1 = \alpha(1, -1)$, $\alpha \in \mathbb{R}$, we obtain

$$\begin{aligned} \alpha(1, -1) \left[\begin{array}{cc} 2c_{11}^E - 1 & 2c_{12}^E \\ 2c_{12}^E & 2c_{11}^E - 1 \end{array} \right] &= \alpha(1, -1) \left[\begin{array}{cc} 2c_{11}^E - 1 & 2(1 - c_{11}^E) \\ 2(1 - c_{11}^E) & 2c_{11}^E - 1 \end{array} \right] \\ &= \alpha [2c_{11}^E - 1 - 2 + 2c_{11}^E, 2 - 2c_{11}^E - 2c_{11}^E + 1] \\ &= \alpha(1, -1) [4c_{11}^E - 3] \\ &= \alpha(1, -1) \left[4 \left(\sum_{k \in 2\mathbb{N} \cap \Lambda} (c_{11}^k + c_{22}^k) \right) - 3 \right]. \end{aligned}$$

So we can rewrite equation (3.14) as

$$\alpha(1, -1) \left[4 \left(\sum_{\substack{k \text{ even} \\ k \geq 0}} (c_{11}^k + c_{22}^k) \right) - 3 \right] = (1, -1) \left(\sum_{\substack{k \text{ even} \\ k \geq 0}} k(c_{11}^k - c_{22}^k + c_{21}^k - c_{12}^k) \right).$$

Hence, we have

$$\alpha \left[4 \left(\sum_{\substack{k \text{ even} \\ k \geq 0}} (c_{11}^k + c_{22}^k) \right) - 3 \right] = \sum_{\substack{k \text{ even} \\ k \geq 0}} k(c_{11}^k - c_{22}^k + c_{21}^k - c_{12}^k).$$

Similarly, for the odds we can express (3.15) as the equation

$$\alpha \left[4 \left(\sum_{\substack{k \text{ odd} \\ k > 0}} (c_{11}^k + c_{22}^k) \right) - 3 \right] = \sum_{\substack{k \text{ odd} \\ k > 0}} k(c_{11}^k - c_{22}^k + c_{21}^k - c_{12}^k).$$

So gathering the equations for $s = 0$ and $s = 1$, we have the following necessary conditions for accuracy 2:

$$\sum_{\substack{k \text{ even} \\ k \geq 0}} (c_{11}^k + c_{12}^k + c_{21}^k + c_{22}^k) = 1$$

$$\sum_{\substack{k \text{ odd} \\ k > 0}} (c_{11}^k + c_{12}^k + c_{21}^k + c_{22}^k) = 1$$

$$\alpha \left[4 \left(\sum_{\substack{k \text{ even} \\ k \geq 0}} (c_{11}^k + c_{22}^k) \right) - 3 \right] = \sum_{\substack{k \text{ even} \\ k \geq 0}} k(c_{11}^k - c_{22}^k + c_{21}^k - c_{12}^k)$$

$$\alpha \left[4 \left(\sum_{\substack{k \text{ odd} \\ k > 0}} (c_{11}^k + c_{22}^k) \right) - 3 \right] = \sum_{\substack{k \text{ odd} \\ k > 0}} k(c_{11}^k - c_{22}^k + c_{21}^k - c_{12}^k).$$

Here, as in the discussion at the end of Example 3.1, we are looking at the free entries in the coefficient matrices as the entries of the first row of the matrix associated to

0 and the entries of the matrices associated to the positive values in Λ . If we solve these four equations for α and the c_{ij}^k , $k \geq 0$ we will satisfy the necessary conditions for the solution to the refinement equation having accuracy at least 2.

3.3.2 The Smith-Barnwell Equations

We now want to ensure the system will be a reproducing system. A necessary condition for this property is that the system must satisfy the Smith-Barnwell equation:

$$M(\omega)M^*(\omega) + M(\omega + \frac{1}{2})M^*(\omega + \frac{1}{2}) = I \quad (3.16)$$

where

$$M(\omega) = \frac{1}{2} \sum_{k \in \Lambda} c^k e^{-2\pi i k \omega} \quad \text{and} \quad M^*(\omega) = \frac{1}{2} \sum_{j \in \Lambda} (c^j)^* e^{2\pi i j \omega}.$$

First we observe that

$$M(\omega)M^*(\omega) = \frac{1}{4} \sum_{k \in \Lambda} \sum_{j \in \Lambda} c^k (c^j)^* e^{-2\pi i (k-j)\omega}$$

and

$$\begin{aligned} M(\omega + \frac{1}{2})M^*(\omega + \frac{1}{2}) &= \frac{1}{4} \sum_{k \in \Lambda} \sum_{j \in \Lambda} c^k (c^j)^* e^{-2\pi i (k-j)\omega} e^{-\pi i (k-j)} \\ &= \frac{1}{4} \sum_{k \in \Lambda} \sum_{j \in \Lambda} c^k (c^j)^* e^{-2\pi i (k-j)\omega} (-1)^{(k-j)}. \end{aligned}$$

Therefore

$$\begin{aligned} M(\omega)M^*(\omega) + M(\omega + \frac{1}{2})M^*(\omega + \frac{1}{2}) &= \frac{1}{4} \sum_{k \in \Lambda} \sum_{j \in \Lambda} (1 + (-1)^{k-j}) c^k (c^j)^* e^{-2\pi i (k-j)\omega} \\ &= \frac{1}{2} \sum_{(k-j) \in 2\mathbb{Z} \cap \Lambda} c^k (c^j)^* e^{-2\pi i (k-j)\omega} \end{aligned}$$

$$= \frac{1}{2} \sum_{n \in \mathbb{Z}} \sum_{k \in \Lambda} c^k (c^{k-2n})^* e^{-2\pi i(2n)\omega}.$$

Now, for a fixed k and a fixed n , we have

$$\begin{aligned} c^k (c^{k-2n})^* &= \begin{pmatrix} c_{11}^k & c_{12}^k \\ c_{21}^k & c_{22}^k \end{pmatrix} \begin{pmatrix} c_{11}^{k-2n} & c_{21}^{k-2n} \\ c_{12}^{k-2n} & c_{22}^{k-2n} \end{pmatrix} \\ &= \begin{pmatrix} c_{11}^k c_{11}^{k-2n} + c_{12}^k c_{12}^{k-2n} & c_{11}^k c_{21}^{k-2n} + c_{12}^k c_{22}^{k-2n} \\ c_{21}^k c_{11}^{k-2n} + c_{22}^k c_{12}^{k-2n} & c_{21}^k c_{21}^{k-2n} + c_{22}^k c_{22}^{k-2n} \end{pmatrix}. \end{aligned}$$

Then, keeping n fixed and summing over $k \in \Lambda$, we can simplify the matrix using the relations (3.3) $c_{11}^k = c_{22}^{-k}$ and $c_{12}^k = c_{21}^{-k}$:

$$\begin{aligned} \sum_{k \in \Lambda} c^k (c^{k-2n})^* &= \sum_{k \in \Lambda} \begin{pmatrix} c_{11}^k c_{11}^{k-2n} + c_{12}^k c_{12}^{k-2n} & c_{11}^k c_{21}^{k-2n} + c_{12}^k c_{22}^{k-2n} \\ c_{21}^k c_{11}^{k-2n} + c_{22}^k c_{12}^{k-2n} & c_{21}^k c_{21}^{k-2n} + c_{22}^k c_{22}^{k-2n} \end{pmatrix} \\ &= \sum_{k \in \Lambda} \begin{pmatrix} c_{11}^k c_{11}^{k-2n} + c_{12}^k c_{12}^{k-2n} & c_{11}^k c_{12}^{-k+2n} + c_{12}^k c_{11}^{-k+2n} \\ c_{12}^{-k} c_{11}^{k-2n} + c_{11}^{-k} c_{12}^{k-2n} & c_{12}^{-k} c_{12}^{-k+2n} + c_{11}^{-k} c_{11}^{-k+2n} \end{pmatrix} \\ &= \sum_{k \in \Lambda} \begin{pmatrix} c_{11}^k c_{11}^{k-2n} + c_{12}^k c_{12}^{k-2n} & c_{11}^{k-2n} c_{12}^{-k} + c_{12}^{k-2n} c_{11}^{-k} \\ c_{12}^{-k} c_{11}^{k-2n} + c_{11}^{-k} c_{12}^{k-2n} & c_{12}^{-k-2n} c_{12}^{-k} + c_{11}^{-k-2n} c_{11}^{-k} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k \in \Lambda} (c_{11}^k c_{11}^{k-2n} + c_{12}^k c_{12}^{k-2n}) & \sum_{k \in \Lambda} (c_{11}^{k-2n} c_{12}^{-k} + c_{12}^{k-2n} c_{11}^{-k}) \\ \sum_{k \in \Lambda} (c_{12}^{-k} c_{11}^{k-2n} + c_{11}^{-k} c_{12}^{k-2n}) & \sum_{k \in \Lambda} (c_{12}^{-k-2n} c_{12}^{-k} + c_{11}^{-k-2n} c_{11}^{-k}) \end{pmatrix}. \end{aligned} \tag{3.17}$$

For $l = -k$, we see that

$$\sum_{k \in \Lambda} (c_{11}^{-k} c_{11}^{-k-2n} + c_{12}^{-k} c_{12}^{-k-2n}) = \sum_{l \in \Lambda} (c_{11}^l c_{11}^{l-2n} + c_{12}^l c_{12}^{l-2n}).$$

One final simplification is

$$\begin{aligned}
\sum_{k \in \Lambda} (c_{11}^{k-2n} c_{12}^{-k} + c_{11}^{-k} c_{12}^{k-2n}) &= \sum_{k \in \Lambda} c_{11}^{k-2n} c_{12}^{-k} + \sum_{k \in \Lambda} c_{11}^{-k} c_{12}^{k-2n} \\
&= \sum_{k \in \Lambda} c_{11}^{k-2n} c_{12}^{-k} + \sum_{j \in \Lambda} c_{11}^{j-2n} c_{12}^{-j} \\
&= 2 \sum_{k \in \Lambda} c_{11}^{k-2n} c_{12}^{-k}.
\end{aligned} \tag{3.18}$$

Therefore, we define q_1 and q_2 as follows:

$$\begin{aligned}
q_1(\Lambda, n) &= \sum_{k \in \Lambda} (c_{11}^k c_{11}^{k-2n} + c_{12}^k c_{12}^{k-2n}) \\
q_2(\Lambda, n) &= 2 \sum_{k \in \Lambda} c_{11}^{k-2n} c_{12}^{-k}.
\end{aligned}$$

Therefore, from (3.17) and (3.18), we arrive at

$$\sum_{k \in \Lambda} c^k (c^{k-2n})^* = \begin{pmatrix} q_1(\Lambda, n) & q_2(\Lambda, n) \\ q_2(\Lambda, n) & q_1(\Lambda, n) \end{pmatrix}.$$

Now (3.16) can be written

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &= M(\omega)M^*(\omega) + M(\omega + \frac{1}{2})M^*(\omega + \frac{1}{2}) \\
&= \frac{1}{2} \sum_{n \in \mathbb{Z}} \sum_{k \in \Lambda} c^k (c^{k-2n})^* e^{-2\pi i(2n)\omega} \\
&= \frac{1}{2} \sum_{n \in \mathbb{Z}} \begin{pmatrix} q_1(\Lambda, n) & q_2(\Lambda, n) \\ q_2(\Lambda, n) & q_1(\Lambda, n) \end{pmatrix} e^{-2\pi i(2n)\omega}.
\end{aligned}$$

Therefore, the entries of the coefficient matrices must satisfy the following equations:

$$q_1(\Lambda, n) = \sum_{k \in \Lambda} (c_{11}^k c_{11}^{k-2n} + c_{12}^k c_{12}^{k-2n}) = 2\delta_{n,0} \quad (3.19)$$

and

$$q_2(\Lambda, n) = \sum_{k \in \Lambda} c_{11}^{k-2n} c_{12}^{-k} = 0, \quad \forall n \in \mathbb{Z}. \quad (3.20)$$

3.3.3 The Composite Dilation Wavelet

In order to find a specific solution, we choose $\Lambda = \{-2, -1, 0, 1, 2\}$ and continue to use $A = 2$ and $B = \{1, -1\}$. We seek a composite dilation wavelet with accuracy $p = 2$ and compact support. Recalling the property that we can solve the accuracy equation in general whenever $p \leq \min \{4 \lfloor \frac{n+1}{2} \rfloor, 4 \lfloor \frac{n}{2} \rfloor + 2\}$, we are lead to believe that such a solution exists since

$$p = 2 < \min \left\{ 4 \left\lfloor \frac{2+1}{2} \right\rfloor, 4 \left\lfloor \frac{2}{2} \right\rfloor + 2 \right\} = 4.$$

In this discussion, we will not use the technique from Section 3.2, but will compute the entries of the coefficient matrices explicitly. We gather the accuracy equations from Section 3.3.1 and the Smith-Barnwell equations from Section 3.3.2 for the coefficient matrices. In order to more descriptively represent the Smith-Barnwell equation, we break equation (3.20) into two equations, one for $n \geq 0$ and the other for $n > 0$, therefore only considering $n \in \mathbb{N}$. Therefore, the entries of our coefficient matrices must satisfy the following system of equations:

$$\sum_{\substack{k \text{ even} \\ k \geq 0}} (c_{11}^k + c_{12}^k + c_{21}^k + c_{22}^k) = 1$$

$$\sum_{\substack{k \text{ odd} \\ k > 0}} (c_{11}^k + c_{12}^k + c_{21}^k + c_{22}^k) = 1$$

$$\alpha \left[4 \left(\sum_{\substack{k \text{ even} \\ k \geq 0}} c_{11}^k \right) - 3 \right] = \sum_{\substack{k \text{ even} \\ k \geq 0}} k(c_{11}^k - c_{22}^k + c_{21}^k - c_{12}^k)$$

$$\alpha \left[4 \left(\sum_{\substack{k \text{ odd} \\ k > 0}} (c_{11}^k + c_{22}^k) \right) - 3 \right] = \sum_{\substack{k \text{ odd} \\ k > 0}} k(c_{11}^k - c_{22}^k + c_{21}^k - c_{12}^k).$$

$$\sum_{k \in \Lambda} (c_{11}^k c_{11}^{k-2n} + c_{12}^k c_{12}^{k-2n}) = 2\delta_{n,0}$$

$$\sum_{k \in \Lambda} c_{11}^{k-2n} c_{12}^{-k} = 0, \quad \forall n \geq 0$$

$$\sum_{k \in \Lambda} c_{11}^{k+2n} c_{12}^{-k} = 0, \quad \forall n > 0.$$

With $\Lambda = \{-2, \dots, 2\}$, we can write these equations explicitly in terms of the entries of the matrices c^k .

Accuracy:

$$(c_{11}^0 + c_{12}^0 + c_{21}^2 + c_{12}^2 + c_{21}^2 + c_{22}^2) = 1$$

$$(c_{11}^1 + c_{12}^1 + c_{21}^1 + c_{22}^1) = 1$$

$$\alpha (4(c_{11}^0 + c_{21}^2 + c_{22}^2) - 3) = 2(c_{11}^2 - c_{12}^2 + c_{21}^2 - c_{22}^2)$$

$$\alpha (4(c_{11}^1 + c_{22}^1) - 3) = (c_{11}^1 - c_{12}^1 + c_{21}^1 - c_{22}^1)$$

Smith-Barnwell:

n=0:

$$(c_{11}^2)^2 + (c_{11}^1)^2 + (c_{11}^0)^2 + (c_{22}^2)^2 + (c_{22}^1)^2 + (c_{22}^0)^2 + (c_{12}^2)^2 + (c_{12}^1)^2 + (c_{12}^0)^2 + (c_{21}^2)^2 + (c_{21}^1)^2 + (c_{21}^0)^2 = 2$$

$$c_{11}^2 c_{21}^2 + c_{11}^1 c_{21}^1 + c_{11}^0 c_{12}^0 + c_{22}^2 c_{12}^1 + c_{22}^1 c_{12}^2 = 0$$

n=1:

$$c_{11}^2 c_{11}^0 + c_{11}^1 c_{22}^1 + c_{11}^0 c_{22}^2 + c_{12}^2 c_{12}^0 + c_{12}^1 c_{21}^1 + c_{12}^0 c_{21}^2 = 0$$

$$c_{11}^0 c_{21}^2 + c_{22}^1 c_{21}^1 + c_{22}^2 c_{12}^0 = 0$$

$$c_{11}^2 c_{12}^0 + c_{11}^1 c_{12}^1 + c_{11}^0 c_{12}^2 = 0$$

n=2:

$$c_{11}^2 c_{22}^2 + c_{12}^2 c_{21}^2 = 0$$

$$c_{22}^2 c_{21}^2 = 0$$

$$c_{11}^2 c_{12}^2 = 0.$$

One can derive the first Smith-Barnwell equation for $n = 0$, the sum of the squares of the free matrix entries, from the remaining accuracy and Smith-Barnwell equations. Thus, this equation is redundant and adds no value to the system of equations. The final two Smith-Barnwell equations for $n = 2$ allow us to make choices for the location of zeros, i.e. if $c_{11}^2 c_{12}^2 = 0$ then either $c_{11}^2 = 0$ or $c_{12}^2 = 0$. So we make the arbitrary choices $c_{11}^2 = c_{21}^2 = 0$. With these choices and the removal of the redundant equation, in order for a solution of the refinement equation to have accuracy $p = 2$, we must solve

Accuracy:

$$(c_{11}^0 + c_{12}^0 + c_{12}^2 + c_{22}^2) = 1$$

$$(c_{11}^1 + c_{12}^1 + c_{21}^1 + c_{22}^1) = 1$$

$$\alpha (4(c_{11}^0 + c_{22}^2) - 3) = -2(c_{12}^2 + c_{22}^2)$$

$$\alpha (4(c_{11}^1 + c_{22}^1) - 3) = (c_{11}^1 - c_{12}^1 + c_{21}^1 - c_{22}^1)$$

Smith-Barnwell:

$$c_{11}^1 c_{21}^1 + c_{11}^0 c_{12}^0 + c_{22}^1 c_{12}^1 + c_{22}^2 c_{12}^2 = 0$$

$$c_{11}^1 c_{22}^1 + c_{11}^0 c_{22}^2 + c_{12}^2 c_{12}^0 + c_{12}^1 c_{21}^1 = 0$$

$$c_{22}^1 c_{21}^1 + c_{22}^2 c_{12}^0 = 0$$

$$c_{11}^1 c_{12}^1 + c_{11}^0 c_{12}^2 = 0.$$

Including α , this is a system of 8 equations with 9 free variables. Therefore, we can solve this system of equations as a one-parameter family of equations in terms of α . Using MatLab, we get a large continuous family of solutions. When we restrict our attention to finding real matrices, we must isolate the values of α that give us real c_{ij}^k . These values of α are

$$\alpha \in \left[\frac{-5 - \sqrt{19}}{6}, \frac{-5 + \sqrt{19}}{6} \right] \cup \left[\frac{3 - \sqrt{3}}{6}, \frac{3 + \sqrt{3}}{6} \right]. \quad (3.21)$$

Let us take $\alpha = \frac{3 - \sqrt{3}}{6}$. Then we can explicitly solve this system of equations to obtain the coefficient matrices

$$c^0 = \frac{1}{4} \begin{pmatrix} 3 - \sqrt{3} & 0 \\ 0 & 3 - \sqrt{3} \end{pmatrix}, \quad c^1 = \frac{1}{4} \begin{pmatrix} 3 + \sqrt{3} & 1 - \sqrt{3} \\ 0 & 0 \end{pmatrix},$$

$$c^2 = \frac{1}{4} \begin{pmatrix} 0 & 1 + \sqrt{3} \\ 0 & 0 \end{pmatrix}. \quad (3.22)$$

Then the remaining matrices, c^{-1} and c^{-2} , are defined by the relations, namely

$$c^{-1} = \frac{1}{4} \begin{pmatrix} 0 & 0 \\ 1 - \sqrt{3} & 3 + \sqrt{3} \end{pmatrix}, \quad c^{-2} = \frac{1}{4} \begin{pmatrix} 0 & 0 \\ 1 + \sqrt{3} & 0 \end{pmatrix}. \quad (3.23)$$

Now we must find $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\Phi(x) = \begin{pmatrix} \varphi(x) \\ \varphi(-x) \end{pmatrix}$ satisfies the refine-

ment equation $\Phi(x) = \sum_{k \in \Lambda} c^k \Phi(2x - k)$. So

$$\Phi(x) = c^{-2}\Phi(2x + 2) + c^{-1}\Phi(2x + 1) + c^0\Phi(2x) + c^1\Phi(2x - 1) + c^2\Phi(2x - 2).$$

Then

$$\begin{aligned} \begin{pmatrix} \varphi(x) \\ \varphi(-x) \end{pmatrix} &= \frac{1}{4} \left[\begin{pmatrix} 0 & 0 \\ 1 + \sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} \varphi(2x + 2) \\ \varphi(-2x - 2) \end{pmatrix} \right. \\ &\quad + \begin{pmatrix} 0 & 0 \\ 1 - \sqrt{3} & 3 + \sqrt{3} \end{pmatrix} \begin{pmatrix} \varphi(2x + 1) \\ \varphi(-2x - 1) \end{pmatrix} \\ &\quad + \begin{pmatrix} 3 - \sqrt{3} & 0 \\ 0 & 3 - \sqrt{3} \end{pmatrix} \begin{pmatrix} \varphi(2x) \\ \varphi(-2x) \end{pmatrix} \\ &\quad + \begin{pmatrix} 3 + \sqrt{3} & 1 - \sqrt{3} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi(2x - 1) \\ \varphi(-2x + 1) \end{pmatrix} \\ &\quad \left. + \begin{pmatrix} 0 & 1 + \sqrt{3} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi(2x - 1) \\ \varphi(-2x + 2) \end{pmatrix} \right]. \end{aligned}$$

Then

$$\begin{aligned} \varphi(x) &= \frac{1}{4} \left[(3 - \sqrt{3})\varphi(2x) + (3 + \sqrt{3})\varphi(2x - 1) \right. \\ &\quad \left. + (1 - \sqrt{3})\varphi(-2x + 1) + (1 + \sqrt{3})\varphi(-2x + 2) \right] \end{aligned} \quad (3.24)$$

and

$$\begin{aligned} \varphi(-x) &= \frac{1}{4} \left[(1 + \sqrt{3})\varphi(2x + 2) + (1 - \sqrt{3})\varphi(2x + 1) \right. \\ &\quad \left. + (3 + \sqrt{3})\varphi(-2x - 1) + (3 - \sqrt{3})\varphi(-2x) \right]. \end{aligned} \quad (3.25)$$

We clearly see that equations (3.24) and (3.25) are equivalent by substituting $-x$ for x in (3.24). This allows us to consider the single function φ as a refinable function with the given refinement equation. That is, φ is the function to which the iterated function system

$$f(x) = \frac{1}{4} \left[(3 - \sqrt{3})f(2x) + (3 + \sqrt{3})f(2x - 1) \right. \\ \left. + (1 - \sqrt{3})f(-2x + 1) + (1 + \sqrt{3})f(-2x + 2) \right] \quad (3.26)$$

converges. Writing $f_t(x)$ to represent t iterations of (3.26), we see that $\varphi(x) = \lim_{t \rightarrow \infty} f_t(x)$. Using MatLab, we begin with the tent function $f_0(x) = (1 - |x|)\chi_{[-1,1]}$ and iterate the function in equation (3.26). Figures 3.1, 3.2, and 3.3 show these plots for $t = 0, 3, 6$, respectively. We observe that as $t \rightarrow \infty$, $f_t(x)$ appears to be converging to a linear function times the characteristic function of $[0, 1]$.

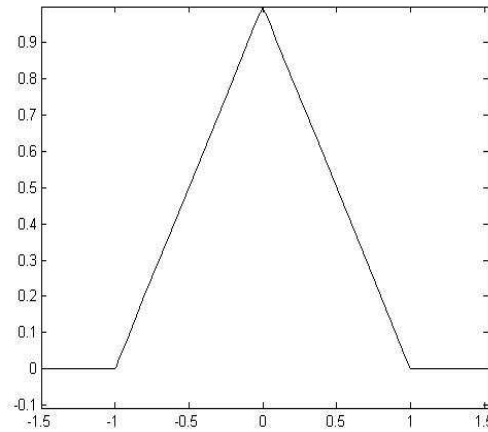


Figure 3.1: The tent function after 0 iterations in (3.26).

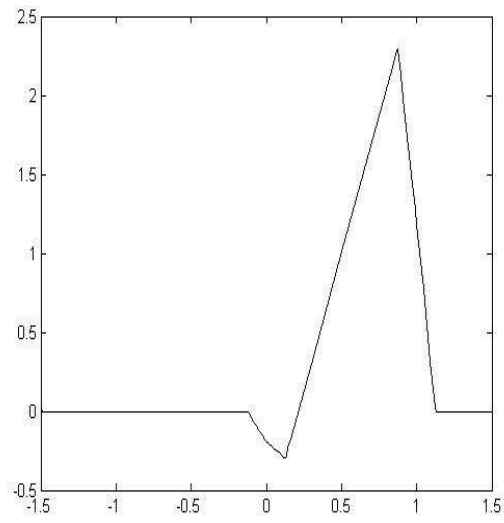


Figure 3.2: The tent function after 3 iterations in (3.26).

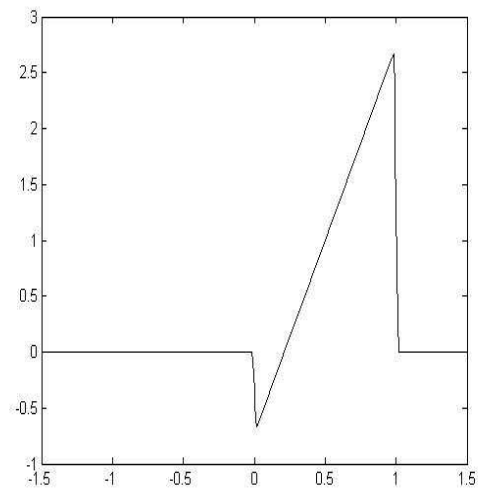


Figure 3.3: The tent function after 6 iterations in (3.26).

Let $\varphi(x) = (\alpha x + \beta)\chi_{[0,1]}$ and solve equation (3.24) for α and β :

$$\begin{aligned}
(\alpha x + \beta)\chi_{[0,1]} &= \frac{1}{4} \left[(3 - \sqrt{3})(\alpha(2x) + \beta)\chi_{[0,1]}(2x) \right. \\
&\quad + (3 + \sqrt{3})(\alpha(2x - 1) + \beta)\chi_{[0,1]}(2x - 1) \\
&\quad + (1 - \sqrt{3})(\alpha(-2x + 1) + \beta)\chi_{[0,1]}(-2x + 1) \\
&\quad \left. + (1 + \sqrt{3})(\alpha(-2x + 2) + \beta)\chi_{[0,1]}(-2x + 2) \right] \\
&= \frac{1}{4} \left[(3 - \sqrt{3})(2\alpha x + \beta)\chi_{[0, \frac{1}{2}]}(x) \right. \\
&\quad + (3 + \sqrt{3})(2\alpha x + \beta - \alpha)\chi_{[\frac{1}{2}, 1]}(x) \\
&\quad + (1 - \sqrt{3})(-2\alpha x + \beta + \alpha)\chi_{[0, \frac{1}{2}]}(x)(-2x + 1) \\
&\quad \left. + (1 + \sqrt{3})(-2\alpha x + \beta + 2\alpha)\chi_{[\frac{1}{2}, 1]}(x) \right] \\
&= \frac{1}{4} \left[(3 - \sqrt{3})(2\alpha x + \beta) + (1 - \sqrt{3})(-2\alpha x + \beta + \alpha) \right] \chi_{[0, \frac{1}{2}]}(x) \\
&\quad + \frac{1}{4} \left[(3 + \sqrt{3})(2\alpha x + \beta - \alpha) + (1 + \sqrt{3})(-2\alpha x + \beta + 2\alpha) \right] \chi_{[\frac{1}{2}, 1]}(x).
\end{aligned} \tag{3.27}$$

So we solve the following system of 2 equations with 2 unknowns:

$$\begin{aligned}
(\alpha x + \beta) &= \frac{1}{4} \left[(3 - \sqrt{3})(2\alpha x + \beta) + (1 - \sqrt{3})(-2\alpha x + \beta + \alpha) \right] \\
(\alpha x + \beta) &= \frac{1}{4} \left[(3 + \sqrt{3})(2\alpha x + \beta - \alpha) + (1 + \sqrt{3})(-2\alpha x + \beta + 2\alpha) \right].
\end{aligned}$$

This leads us to the one parameter family of solutions

$$\beta = \frac{1 - \sqrt{3}}{2\sqrt{3}}\alpha.$$

As a scaled refinable function will satisfy the same refinement equation, then we have

a one parameter family of functions that will satisfy our refinement equation (3.1):

$$\varphi_\alpha(x) = \left(\alpha x + \frac{1 - \sqrt{3}}{2\sqrt{3}}\alpha \right) \chi_{[0,1]}(x). \quad (3.28)$$

In order to normalize the scaling function, φ , we choose $\alpha = \sqrt{6}$ and define

$$\varphi(x) = \varphi_{\sqrt{6}}(x) = \left(\sqrt{6}x + \frac{1 - \sqrt{3}}{\sqrt{2}} \right) \chi_{[0,1]}(x). \quad (3.29)$$

The graph of our scaling function defined by (3.29) appears in figure 3.4.

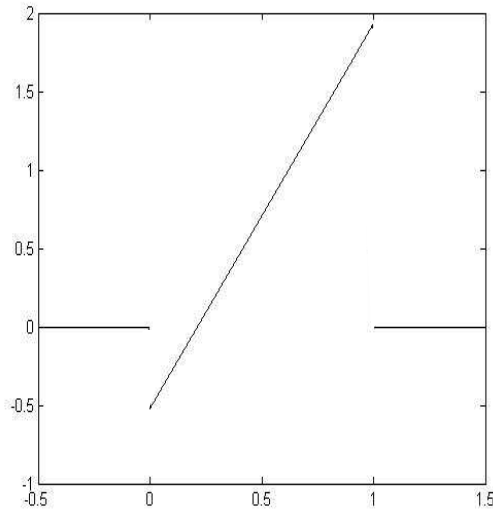


Figure 3.4: The scaling function $\varphi(x)$.

With $\Phi(x) = (\varphi(x), \varphi(-x))^t$ we have a solution to the refinement equation (3.1)

$$\Phi(x) = \sum_{k \in \Lambda} c^k \Phi(2x - k)$$

for $\Lambda = \{-2, \dots, 2\}$ and the c^k which are the solutions (3.22) and (3.23).

From the preceding discussion, this solution has accuracy 2 and satisfies the Smith-Barnwell equation. Also, we chose $\alpha = \sqrt{6}$ so that $\|\varphi\|_2 = 1$. Now we see from the

following straightforward calculation that the translates of $\varphi(x)$ and $D_{-1}\varphi(x)$ are also orthogonal. For any two integers $k, l \in \mathbb{Z}$,

$$\begin{aligned}
& \langle T_k\varphi(x), T_l D_{-1}\varphi(x) \rangle \\
&= \langle \varphi(x-k), \varphi(-(x-l)) \rangle \\
&= \int_{\mathbb{R}} \varphi(x-k)\varphi(l-x)dx \\
&= \int_{\mathbb{R}} \left(\sqrt{6}(x-k) + \frac{1-\sqrt{3}}{\sqrt{2}} \right) \chi_{[0,1]}(x-k) \left(\sqrt{6}(l-x) + \frac{1-\sqrt{3}}{\sqrt{2}} \right) \chi_{[0,1]}(l-x)dx \\
&= \int_{\mathbb{R}} \left(\sqrt{6}x - \sqrt{6}k + \frac{1-\sqrt{3}}{\sqrt{2}} \right) \chi_{[k,k+1]}(x) \left(-\sqrt{6}x + \sqrt{6}l + \frac{1-\sqrt{3}}{\sqrt{2}} \right) \chi_{[l-1,l]}(x)dx \\
&= \delta_{k,l-1} \int_k^{k+1} \left(\sqrt{6}x - \sqrt{6}k + \frac{1-\sqrt{3}}{\sqrt{2}} \right) \left(-\sqrt{6}x + \sqrt{6}l + \frac{1-\sqrt{3}}{\sqrt{2}} \right) dx \\
&= \delta_{k,l-1} \int_k^{k+1} \left(\sqrt{6}x + \frac{1-(2k+1)\sqrt{3}}{\sqrt{2}} \right) \left(-\sqrt{6}x + \frac{1+(2k+1)\sqrt{3}}{\sqrt{2}} \right) dx \\
&= \delta_{k,l-1} \left[-2x^3 + 3(2k+1)x^2 + \frac{1-3(2k+1)^2}{2}x \right] \Big|_k^{k+1} \\
&= \delta_{k,l-1} [-6k^2 - 6k + 6k^2 + 6k] \\
&= 0.
\end{aligned}$$

Therefore, $\{D_b T_k \varphi : b \in \{1, -1\}, k \in \mathbb{Z}\}$ is an orthonormal system with accuracy 2. Furthermore, this orthonormal system generates a shift invariant space that will be the scaling space for our composite dilation MRA, namely $V_0 = \overline{\text{span}}\{D_b T_k \varphi : b \in \{1, -1\}, k \in \mathbb{Z}\}$. To construct the composite dilation MRA, we define $V_j = D_2^{-j} V_0$. Since φ satisfies the refinement equation, or the 2-scale equation, we know that $D_2^{-1} V_0 = V_1 = \overline{\text{span}}\{D_b T_k \varphi(2\cdot) : b \in \{1, -1\}, k \in \mathbb{Z}\}$.

At this point, we want to find the composite dilation wavelet $\Psi = (\psi, D_{-1}\psi)^t$ associated with the scaling function $\Phi = (\varphi, D_{-1}\varphi)^t$. We seek a function $\psi \in V_1$ such that $\overline{\text{span}}\{D_b T_k \psi : b \in \{1, -1\}, k \in \mathbb{Z}\} = W_0$ where $V_0 \perp W_0$ and $V_0 \oplus W_0 = V_1$. For such a function to exist with support on $[0, 1]$, there must exist α, β, γ , and δ (α and

β are different unknowns than from the family of solutions (3.28)) such that

$$\begin{aligned}\psi(x) &= \alpha\varphi(2x) + \beta D_{-1}\varphi(2x-1) + \gamma\varphi(2x-1) + \delta D_{-1}\varphi(2x-2) \\ &= \alpha\varphi(2x) + \beta\varphi(1-2x) + \gamma\varphi(2x-1) + \delta\varphi(2-2x).\end{aligned}$$

If $V_0 \perp W_0$, then $\langle T_k\psi, T_l\varphi \rangle = \langle T_k\psi, T_l D_{-1}\varphi \rangle = 0$ for any $k, l \in \mathbb{Z}$. Also, if $\{D_b T_k\psi\}$ is to be an orthogonal system, then $\langle T_k\psi, T_l D_{-1}\psi \rangle = 0$ for all $k, l \in \mathbb{Z}$. Therefore, we can solve these orthogonality conditions to obtain solutions for α, β, γ , and δ . (From the discussion of the orthogonality of the translates of φ and $D_{-1}\varphi$, we see that the $\delta_{k,l-1}$ allow us to simply check the orthogonality conditions when $k=0$ and $l=1$.)

Using the facts that $\varphi(2\cdot)$ and $\varphi(1-2\cdot)$ are supported only on $[0, \frac{1}{2}]$ and $\varphi(2\cdot-1)$ and $\varphi(2-2\cdot)$ are supported only on $[\frac{1}{2}, 1]$, we find a system of equations for the unknown scalars.

$$\begin{aligned}\langle \psi, \varphi \rangle &= \int_{\mathbb{R}} \psi(x)\varphi(x)dx \\ &= \int_0^{\frac{1}{2}} (\alpha\varphi(x)\varphi(2x) + \beta\varphi(x)\varphi(1-2x)) dx \\ &\quad + \int_{\frac{1}{2}}^1 (\gamma\varphi(x)\varphi(2x-1) + \delta\varphi(x)\varphi(2-2x)) dx \\ &= \frac{1}{8} \left[(3 - \sqrt{3})\alpha + (1 - \sqrt{3})\beta + (3 + \sqrt{3})\gamma + (1 + \sqrt{3})\delta \right].\end{aligned}$$

Similarly we obtain

$$\begin{aligned}\langle \psi, T_1 D_{-1}\varphi \rangle &= \int_{\mathbb{R}} \psi(x)\varphi(1-x)dx \\ &= \frac{1}{8} \left[(1 + \sqrt{3})\alpha + (3 + \sqrt{3})\beta + (1 - \sqrt{3})\gamma + (3 - \sqrt{3})\delta \right]\end{aligned}$$

$$\langle \psi, T_1 D_{-1}\psi \rangle = \int_{\mathbb{R}} \psi(x)\psi(1-x)dx$$

$$= \frac{1}{2} [\alpha\delta + \beta\gamma].$$

Therefore, to find appropriate values for $\alpha, \beta, \gamma, \delta$ we solve the system of equations

$$\begin{aligned} (3 - \sqrt{3})\alpha + (1 - \sqrt{3})\beta + (3 + \sqrt{3})\gamma + (1 + \sqrt{3})\delta &= 0 \\ (1 + \sqrt{3})\alpha + (3 + \sqrt{3})\beta + (1 - \sqrt{3})\gamma + (3 - \sqrt{3})\delta &= 0 \\ \alpha\delta + \beta\gamma &= 0. \end{aligned}$$

From these equations we get another one-parameter family:

$$\begin{aligned} \alpha &= \alpha & (3.30) \\ \beta &= \sqrt{3}\alpha \\ \gamma &= (2 + \sqrt{3})\alpha \\ \delta &= -(3 + 2\sqrt{3})\alpha. \end{aligned}$$

Each of these solutions will guarantee orthogonality. For ease of computations, we will choose $\alpha = 1$ for the following calculations and normalize the wavelet later. We now have a compactly supported scaling function φ and an associated compactly supported orthogonal wavelet, $\tilde{\psi}$, where

$$\tilde{\psi} = \varphi(2x) + \sqrt{3}\varphi(1 - 2x) + (2 + \sqrt{3})\varphi(2x - 1) - (3 + \sqrt{3})\varphi(2 - 2x).$$

Since $\varphi(2\cdot)$ and $\varphi(1 - 2\cdot)$ are supported on $[0, \frac{1}{2}]$ and $\varphi(2\cdot - 1)$ and $\varphi(2 - 2\cdot)$ are supported on $[\frac{1}{2}, 1]$, then $\tilde{\psi}$ is supported on $[0, 1]$. Therefore, as expected, the scaling function φ and the wavelet $\tilde{\psi}$ have the same support. Consequently, $D_{-1}\varphi(x) = \varphi(-x)$ and $D_{-1}\tilde{\psi}(x) = \tilde{\psi}(-x)$ also have the same support, $[-1, 0]$. Thus, $\Phi = (\varphi, D_{-1}\varphi)^t$ and $\tilde{\Psi} = (\tilde{\psi}, D_{-1}\tilde{\psi})^t$ have precisely the same support. Since Φ and $\tilde{\Psi}$ are

orthogonal, as shown above, we know $V_0 \perp W_0$.

We now must show that $V_0 \oplus W_0 = V_1 = D_2^{-1}V_0$. It is possible to show this using the dimension function for the shift invariant spaces. However, in the spirit of the above calculations, we show this directly. Since $V_1 = \overline{\text{span}}\{D_b T_k \varphi(2\cdot) : b \in \{1, -1\}, k \in \mathbb{Z}\}$, if we are able to construct $\varphi(2\cdot)$ from a linear combination of translations of φ , $D_{-1}\varphi$, $\tilde{\psi}$, and $D_{-1}\tilde{\psi}$, then we clearly see that $V_1 = V_0 \oplus W_0$. Therefore, let

$$\varphi(2x) = a\varphi(x) + b\varphi(1-x) + c\tilde{\psi}(x) + d\tilde{\psi}(1-x).$$

Expanding $\tilde{\psi}$ in terms of φ we obtain the equation

$$\begin{aligned} \varphi(2x) &= a\varphi(x) + b\varphi(1-x) + \left(c - (3 + 2\sqrt{3})d\right) \varphi(2x) \\ &\quad + \left(\sqrt{3}c + (2 + \sqrt{3})d\right) \varphi(1-2x) + \left((2 + \sqrt{3})c + \sqrt{3}d\right) \varphi(2x-1) \\ &\quad + \left(d - (3 + 2\sqrt{3})c\right) \varphi(2-2x). \end{aligned}$$

Again, since $\varphi(2x)$ is supported on $[0, \frac{1}{2}]$, we get the following two equations:

$$\begin{aligned} \varphi(2x) &= [a\varphi(x) + b\varphi(1-x)] \chi_{[0, \frac{1}{2}]}(x) + \left(c - (3 + 2\sqrt{3})d\right) \varphi(2x) \\ &\quad + \left(\sqrt{3}c + (2 + \sqrt{3})d\right) \varphi(1-2x) \\ 0 &= [a\varphi(x) + b\varphi(1-x)] \chi_{[\frac{1}{2}, 1]}(x) + \left((2 + \sqrt{3})c + \sqrt{3}d\right) \varphi(2x-1) \\ &\quad + \left(d - (3 + 2\sqrt{3})c\right) \varphi(2-2x). \end{aligned}$$

Substituting $\varphi(x) = \left(\sqrt{6}x + \frac{1-\sqrt{3}}{\sqrt{2}}\right) \chi_{[0,1]}(x)$ into the equation, equating the linear and constant coefficients, and dividing by appropriate scalars we obtain the following four equations:

$$a - b + 2 \left((1 - \sqrt{3})c - (5 + 3\sqrt{3})d \right) = 2$$

$$\begin{aligned}
a - b + 2 \left((5 + 3\sqrt{3})c - (1 - \sqrt{3})d \right) &= 0 \\
a - (2 + \sqrt{3})b - (2 + 2\sqrt{3})c - (10 + 6\sqrt{3})d &= 1 \\
a - (2 + \sqrt{3})b + (38 + 22\sqrt{3})c - (2 - 2\sqrt{3})d &= 0.
\end{aligned}$$

This system of four linear equations with four unknowns has the explicit solution

$$a = \frac{3 - \sqrt{3}}{8}, \quad a = \frac{1 + \sqrt{3}}{8}, \quad c = \frac{2 - \sqrt{3}}{16}, \quad d = -\frac{\sqrt{3}}{16}.$$

Since $\varphi(2x)$ is the generating function for V_1 , $\varphi(x)$ is the generating function for V_0 , and $\tilde{\psi}(x)$ is the generating function for W_0 , we have established that $V_1 = V_0 \oplus W_0$ as

$$\varphi(2x) = \frac{3 - \sqrt{3}}{8}\varphi(x) + \frac{1 + \sqrt{3}}{8}\varphi(1 - x) + \frac{2 - \sqrt{3}}{16}\tilde{\psi}(x) - \frac{\sqrt{3}}{16}\tilde{\psi}(1 - x).$$

Finally, to generate an orthonormal wavelet basis, we must normalize $\tilde{\psi}$. In order to accomplish this, we utilize the one-parameter family defined by (3.30) and take $\alpha = \frac{1}{\sqrt{8 + \sqrt{13}}}$. Then

$$\begin{aligned}
\psi(x) &= \frac{1}{\sqrt{8 + \sqrt{13}}}\tilde{\psi}(x) \\
&= \frac{1}{\sqrt{8 + \sqrt{13}}}\left[\varphi(2x) + \sqrt{3}\varphi(1 - 2x) + (2 + \sqrt{3})\varphi(2x - 1) - (3 + \sqrt{3})\varphi(2 - 2x)\right].
\end{aligned} \tag{3.31}$$

The graph of our wavelet defined by (3.31) appears in figure 3.5.

Since φ is compactly supported and refinable, then ψ is compactly supported and satisfies the same refinement equation (3.1). Therefore the system $\{D_b T_k \psi : b \in \{1, -1\}, k \in \mathbb{Z}\}$ is a compactly supported, refinable system with accuracy 2. Therefore, ψ is a compactly supported, orthonormal, MRA, composite dilation wavelet on $L^2(\mathbb{R})$ with accuracy 2.

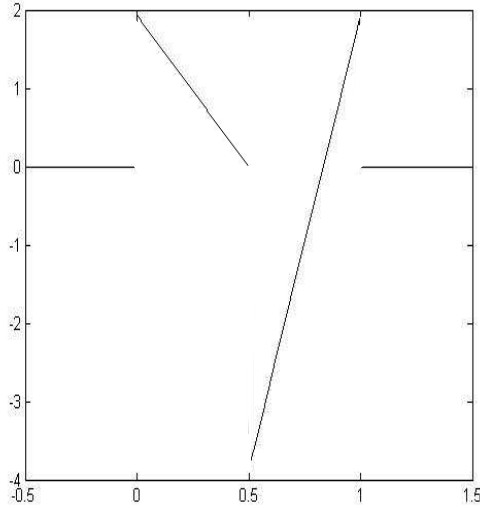


Figure 3.5: The wavelet $\psi(x)$.

A final comment about the results of this process is that this was all done directly using the accuracy equations and Smith-Barnwell equations in terms of the entries of the coefficient matrices. What is interesting, and somewhat surprising, is that the entries of the coefficient matrices that we found are precisely the coefficients of the refinement equation for the Daubechies wavelet with minimal support, usually called D4. The D4 scaling function satisfies the refinement equation

$$f(x) = \frac{1 + \sqrt{3}}{4}f(2x) + \frac{3 + \sqrt{3}}{4}f(2x - 1) + \frac{3 - \sqrt{3}}{4}f(2x - 2) + \frac{1 - \sqrt{3}}{4}f(2x - 3).$$

The scaling function for our system satisfies the refinement equation

$$\varphi(x) = \frac{3 - \sqrt{3}}{4}\varphi(2x) + \frac{3 + \sqrt{3}}{4}\varphi(2x - 1) + \frac{1 - \sqrt{3}}{4}\varphi(-2x + 1) + \frac{1 + \sqrt{3}}{4}\varphi(-2x + 2).$$

This is likely due to the dominant role of the Smith-Barnwell equation in formulating the refinement equations. However, every refinement equation whose solution satisfies the Smith-Barnwell equation does not have the same coefficients. The D4

scaling function is one of a continuous family of scaling functions supported on an interval of length three. It is extremal in this family in that it has the largest possible number of vanishing moments, i.e. it is the smoothest. We also believe that our scaling function is a member of a large family of scaling functions defined by the parameter α (3.21). Our scaling function is obviously extremal in this family as $\alpha = \frac{3-\sqrt{3}}{6}$ is the left end point of one of the intervals from which α produces real-valued solutions to the refinement equation. From (3.22) and (3.22), we see that the coefficient matrices of the refinement equation defining our scaling function are rather sparse. It is likely that our scaling function is extremal in its family in the sense that it has the maximal number of zeros in the coefficient matrices.

Chapter 4

Conclusions

In this dissertation we have answered some of the open questions regarding the existence and accuracy of composite dilation wavelets. This relatively new subset of the theory of wavelets has more questions than answers. In the process of answering the questions addressed in this document, we have generated several new questions.

In Chapter 2, we proved that MSF wavelets exist in every dimension. We did so by examining two admissibility conditions. In theorem 2.2, we showed that an arbitrary lattice and a finite group with fundamental region bounded by hyperplanes through the origin can generate the necessary sets to support a scaling function. A natural question that needs to be answered is whether or not this property of the fundamental region is necessary. If so, the finite groups used for composite dilations will be classified for MSF composite dilation wavelets. In a similar spirit, the next step is to classify the MSF composite dilation wavelets arising from finite groups.

Theorem 2.9 proved that $2I_n$ is (B, Γ) admissible in every dimension. As a result, we were able to show in theorem 2.10 that MRA, MSF composite dilation wavelets exists in every dimension. The number of wavelet generating functions required when using $2I_n$ is $2^n - 1$. This grows far too rapidly to be useful in applications in high dimensions. We then showed that we can reduce the number of wavelet generating

functions to 1 at the expense of losing the freedom to choose our composite dilation group and the lattice. It is important to investigate if one can find a more general reduction in the determinant of the expanding matrix. It is unlikely that we will proceed to higher dimensions at no cost, but it is plausible that we should be able to reduce the number of generators to polynomial growth in dimension rather than exponential growth in dimension.

At the conclusion to the proof of theorem 2.11, we discussed a potential advantage of composite dilation wavelets. Here, we showed that in every dimension we can find an MSF composite dilation wavelet with a single wavelet generating function. This required us to have the order of the composite dilation group, B , grow to the dimension of the space. In applications, this would result in a significant reduction in input requirements for any implementation algorithm. It is important that we study the possibilities in exploiting the group action in implementing composite dilation wavelets.

There is some movement toward using MSF wavelets in applications. For example, shearlets are MSF composite dilation wavelets [8]. The MSF wavelets described in this dissertation generate very coarse orthonormal bases. To be useful in applications, one would need to develop powerful smoothing techniques. In traditional wavelet literature, much has been studied on this approach. Now we need to find similar results for composite dilation wavelets.

In chapter 3, we found the necessary conditions for compactly supported composite dilation wavelets for $L^2(\mathbb{R})$ with accuracy p and exhibited the first known composite dilation wavelet with accuracy greater than the Haar type composite dilation wavelets [13]. This dissertation simply leads us to believe that we can succeed in finding smoother composite dilation wavelets but has taken only small steps in that direction. Much work is left to be done. As the dimension increases, the results in [2] and the necessary equations from the Smith-Barnwell equations create a large set of quadratic

equations that must be solved. Is this direct approach the best way to find these compactly supported functions or is there a better way? Even in one dimension, there are significant obstacles to solving the necessary equations. Can we find a Daubechies-like classification of compactly supported composite dilation wavelets? Can we extend this to even two dimensions?

The success and rapid growth of the field of wavelet analysis is largely due to the vast applications of wavelets. While mathematically beautiful, for composite dilation wavelets to catch on with the same fervor, we will have to find their advantages in applications. Exploiting the group action of the composite dilations and the ability to rotate and reflect the orientation of the wavelets gives considerable promise to the application of composite dilation wavelets. Some significant effort in the application of these orthonormal bases will go a long way toward advancing the subject.

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