MINIMALLY SUPPORTED FREQUENCY COMPOSITE DILATION PARSEVAL FRAME WAVELETS

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ABSTRACT. A composite dilation Parseval frame wavelet is a collection of functions generating a Parseval frame for $L^2(\mathbb{R}^n)$ under the actions of translations from a full rank lattice and dilations by products of elements of groups A and B. A minimally supported frequency composite dilation Parseval frame wavelet has generating functions whose Fourier transforms are characteristic functions of sets contained in a lattice tiling set. Constructive proofs are used to establish the existence of minimally supported frequency composite dilation Parseval frame wavelets in arbitrary dimension using any finite group B, any full rank lattice, and an expanding matrix generating the group A and normalizing the group B. Moreover, every such system is derived from a Parseval frame multiresolution analysis. Multiple examples are provided including examples that capture directional information.

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1 Introduction

The usefulness of frames in mathematics and its applications is well-known and well documented. (For example, the reader could begin with the 294 documents referenced in [7].) One set of frames that receives considerable attention is the family of wavelet frames; frames generated by a single function under the actions of dilations and translations by specified parameters. In wavelet theory, much has been studied on minimally supported frequency (MSF) wavelets, also known as wavelet set wavelets (for example [1], [12], [21], etc.). In turn, much work has been accomplished on minimally supported frequency wavelet frames, or wavelet frames whose support in the Fourier domain is contained in some wavelet set ([4], [7], [11], [10], etc.). MSF frame wavelets provide valuable insight to the theory of frame wavelets but must be smoothed to have adequate time localization for applications.

Recently there has been significant interest in generating frames which capture directional information as well as time and frequency information. Some examples of these oriented oscillatory waveforms are brushlets [24], contourlets [13], curvelets [6], ridgelets [5], and shearlets [16]. Of these only shearlets are examples of an affine systems generated by composite dilations, i.e. shearlets are generated using dilations by integer powers of an expanding matrix and dilations by integer powers of a shear matrix.

Guo, Labate, Lim, Weiss, and Wilson [17, 18, 19] introduced the theory of composite dilation wavelets and detailed the extension of a multiresolution analysis (MRA) to this setting. Here, our primary interest is determining which finite dilation groups, B, and which translation lattices, Γ, give rise to MRA, MSF, composite dilation Parseval frame wavelets. This establishes a foundation for potentially developing smoother directional representation systems with finite groups. These systems will likely be similar to shearlets in that smoothing functions might be applied to the generators of the MSF composite dilation Parseval frame wavelet.

The paper is organized as follows: the remainder of this section provides the necessary background and notation. Section 2 proves the existence of MRA, MSF composite dilation Parseval frame wavelets using any finite group and any full rank lattice. MRA, MSF composite dilation wavelets generating an orthonormal basis for $L^2(\mathbb{R}^n)$ were developed in [3]. Section 3 establishes that these constructions will also produce singly generated MRA, MSF composite dilation Parseval frame wavelets. Section 4 concludes the paper with three examples showing the geometric versatility of this type of Parseval frame.

1.1 Parseval Frames and Composite Dilation Wavelets

A system $\{\phi_i\}_{i=1}^{\infty}$ is a *frame* for $L^2(\mathbb{R}^n)$ if there exist constants $0 < C_1 \leq C_2 < \infty$ (C_1 and C_2 are the *frame bounds*) such that $C_1 \|f\|_2^2 \leq \sum_{i=1}^{\infty} |\langle f, \phi_i \rangle|^2 \leq C_2 \|f\|_2^2$. If $C_1 = C_2$ it is called a *tight frame* and when $C_1 = C_2 = 1$ we have a *Parseval frame*. Obviously, Parseval frames satisfy Parseval's identity $\|f\|_2^2 = \sum_{i=1}^{\infty} |\langle f, \phi_i \rangle|^2$ for all $f \in L^2(\mathbb{R}^n)$.

A linear transformation of the integer lattice is a *full rank lattice*, $\Gamma = c\mathbb{Z}^n$ for $c \in GL_n(\mathbb{R})$, the space of invertible $n \times n$ matrices. A *lattice basis* for $\Gamma = c\mathbb{Z}^n$ is a set $\{\gamma_i\}_{i=1}^n$ such that every element of the lattice, $\gamma \in \Gamma$, can be uniquely written as an integer combination of the basis elements: $\gamma = \sum_{i=1}^n m_i \gamma_i$ where $m_i \in \mathbb{Z}$. Notice that if c_i is the ith column of the matrix c, then the set $\{c_i\}_{i=1}^n$ is a lattice basis for Γ.

For any $k \in \Gamma$, the *translation of f by k*, T_k, acts on a function f in $L^2(\mathbb{R}^n)$ by $T_k f(x) = f(x-k)$. When $a \in GL_n(\mathbb{R})$, the *dilation of f by a* is the operator $D_a f(x) = |\det(a)|^{-\frac{1}{2}} f(a^{-1}x)$. These two unitary operators are applied to a set of generating functions to develop an affine system.

For a countable subset of invertible matrices $C \subset GL_n(\mathbb{R})$, Γ a full rank lattice, and $\psi^1, \ldots, \psi^L \in L^2(\mathbb{R}^n)$, the *affine system produced by C, Γ, and* $\Psi = (\psi^1, \ldots, \psi^L)$ is the set $\mathcal{A}_{C\Gamma}(\Psi) = \left\{D_c T_k \psi^l : c \in C, k \in \Gamma, 1 \leq l \leq L\right\}$. As introduced by Guo, Labate, Lim, Weiss, and Wilson [17, 18, 19] affine systems with composite dilations,

$$\mathcal{A}_{AB\Gamma}(\Psi) = \left\{D_a D_b T_k \psi^l : a \in A, b \in B, k \in \Gamma, 1 \leq l \leq L\right\}, \tag{1}$$

are obtained when $C = AB$ is the product of two not necessarily commuting subsets of invertible matrices. In this discussion, $A = \left\{a^j : j \in \mathbb{Z}\right\}$ for an expanding matrix $a \in GL_n(\mathbb{R})$ while B is a finite subgroup of $GL_n(\mathbb{R})$.

Definition 1. $\Psi = (\psi^1, \ldots, \psi^L) \subset L^2(\mathbb{R}^n)$ is a *composite dilation Parseval frame wavelet* if there exist A, B, and Γ such that $\mathcal{A}_{AB\Gamma}(\Psi)$ is a Parseval frame for $L^2(\mathbb{R}^n)$.

A natural extension of an MSF Parseval frame wavelet [21], we say that Ψ is a *minimally supported frequency composite dilation Parseval frame wavelet* when $\Psi = \left(\psi^1, \ldots, \psi^L\right) \subset L^2(\mathbb{R}^n)$ is an MSF composite dilation Parseval frame wavelet and $\hat{\psi}^l = |\det(c)|^{\frac{1}{2}} \chi_{R_l}$ for some disjoint, measurable sets $R_1, \ldots, R_L \subset \hat{\mathbb{R}}^n$. In order to generate a Parseval frame, each set R_l must be contained in a lattice tiling set of $\hat{\mathbb{R}}^n$ for the full rank lattice associated to the composite dilation system.

We adopt the notation that the time domain is represented by $\mathbb{R}^n$, and its elements will be column vectors denoted by letters of the Roman alphabet, $x = (x_1, \dots, x_n)^t \in \mathbb{R}^n$. The elements of the frequency domain, $\hat{\mathbb{R}}^n$, will be row vectors, $\xi = (\xi_1, \dots, \xi_n) \in \hat{\mathbb{R}}^n$, denoted by letters of the Greek alphabet.

We use the Fourier transform $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi x} dx$. The inverse Fourier transform of $g \in L^2(\hat{\mathbb{R}}^n)$ is $\check{g}$. For $a \in GL_n(\mathbb{R})$ and $g \in L^2(\hat{\mathbb{R}}^n)$, the *Fourier domain dilation of g by a* is the operator $\hat{D}_a g(\xi) = |\det(a)|^{\frac{1}{2}} g(\xi a)$. For $k \in \Gamma$ and $g \in L^2(\hat{\mathbb{R}}^n)$, the *modulation of g by k* is the operator $M_k g(\xi) = e^{-2\pi i \xi k} g(\xi)$. These operators allow us to easily take the Fourier transform of an element of an affine system with composite dilations. Namely, if $a, b \in GL_n(\mathbb{R})$, $k \in \Gamma$, and $j \in \mathbb{Z}$, then $\left[D_a^j D_b T_k f\right]\hat{}(\xi) = \hat{D}_a^j \hat{D}_b M_k \hat{f}(\xi)$.

The full rank lattice $\Gamma^* = \hat{\mathbb{Z}}^n c^{-1}$ is the *dual lattice to* $\Gamma = c\mathbb{Z}^n$. The rows of the matrix c^{-1} form a basis for the dual lattice Γ^*. The notion of a dual lattice is used extensively when defining Parseval frames for subspaces of $L^2(\mathbb{R}^n)$.

1.2 Composite Dilation Multiresolution Analysis

One classical method for constructing Parseval frame wavelets is the well known Parseval frame multiresolution analysis. Guo et al. extend the classical MRA to affine systems with composite dilations [19]. Interested in constructing Parseval frames, we establish the following definition:

Definition 2. Let $a \in GL_n(\mathbb{R})$ be an expanding matrix, $B \subset GL_n(\mathbb{R})$ a finite group, and Γ a full rank lattice. An $AB\Gamma$ *Parseval frame multiresolution analysis (MRA)* is a sequence, $\{V_j\}_{j\in\mathbb{Z}}$, of closed subspaces of $L^2(\mathbb{R}^n)$ satisfying

$$\begin{aligned} (i)&\quad D_b T_k V_0 = V_0, \text{for any } b \in B, k \in \Gamma;\\ (ii)&\quad \text{for each } j \in \mathbb{Z}, V_j \subset V_{j+1} \text{ where } V_j = D_a^{-j} V_0;\\ (iii)&\quad \bigcap_{j\in\mathbb{Z}} V_j = \{0\} \text{ and } \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n);\\ (iv)&\quad \exists\, \varphi \in V_0 \text{ such that } \{D_b T_k \varphi : b \in B, k \in \Gamma\} \text{ is a Parseval frame for } V_0. \end{aligned}$$

Notice that in this setting, the scaling space V_0 has a Parseval frame generated by both dilations from the finite group B and translations from the lattice Γ.

1.3 Group Actions on $\mathbb{R}^n$

The expression $Q_1 \cap Q_2 = \emptyset$ and the term *disjoint* are meant in the sense of measure.

We say a set Q is *bounded by a collection of hyperplanes* $\{H_1, \dots, H_s\}$ when Q is the nonempty intersection of the half-spaces defined by the collection $\{H_1, \dots, H_s\}$.

Definition 3. Let G be a group acting from the right on a measurable set $\Omega \subset \hat{\mathbb{R}}^n$. Then a *$G$-tiling set for* Ω is a measurable set R such that

(i) $\bigcup_{g\in G} Rg = \Omega$ and

(ii) $Rg_1 \cap Rg_2 = \emptyset$ for $g_1 \neq g_2 \in G$.

A set $F \subset \hat{\mathbb{R}}^n$ is a *fundamental region for B* if F is B-tiling set for $\hat{\mathbb{R}}^n$.

Example 1. Let $B = D_4$, the group of symmetries of the square acting on $\hat{\mathbb{R}}^2$; D_4 is the group generated by r_1 and r_2, the reflections through the lines $\xi_2 = \xi_1$ and $\xi_2 = 0$ respectively. Let $F = \{(\xi_1, \xi_2) : \xi_2 \leq \xi_1\} \cap \{(\xi_1, \xi_2) : \xi_2 \geq 0\}$. Then F is bounded by the hyperplanes $H_1 = \{(\xi_1, \xi_2) : \xi_2 = \xi_1\}$ and $H_2 = \{(\xi_1, \xi_2) : \xi_2 = 0\}$. If we let $B = D_4$ act on F, we see that F is a fundamental region for B. See the left portion of Figure 1.

Now let S be the unit square centered at the origin, $S = \left\{(\xi_1, \xi_2) : -\frac{1}{2} \leq \xi_1, \xi_2 \leq \frac{1}{2}\right\}$. Let R be the triangle with vertices $(0,0), (\frac{1}{2}, 0)$, and $(\frac{1}{2}, \frac{1}{2})$. When B acts on R, we see that R is a B-tiling set for S. See the right portion of Figure 1.

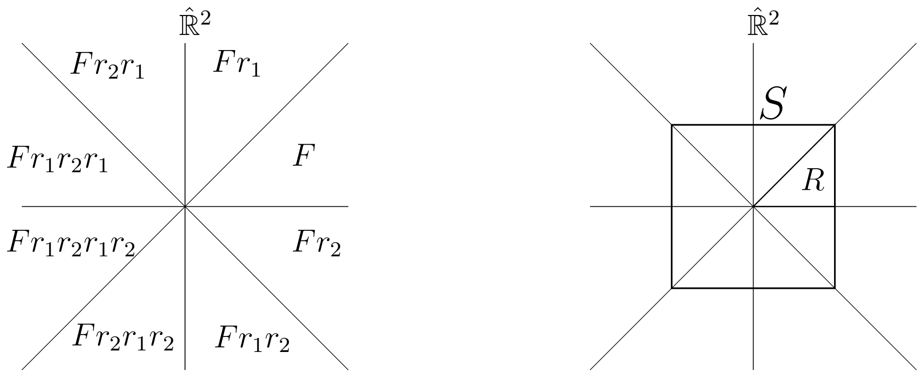

Figure 1: F is a fundamental region for $B = D_4$ (left) and R is a B-tiling set for S (right).

A set S is said to be *starlike with respect to the origin* if, for every $\xi \in S$, the line segment joining the origin and ξ is contained in S. A *Starlike Neighborhood of the origin* is a set S that is both starlike with respect to the origin and contains an open neighborhood of the origin; $N_\epsilon(\vec{0}) \subset S$ for some $\epsilon > 0$.

Throughout this discussion it will often be advantageous to select a lattice Γ that will complement the composite group B in some way. Frequently, we choose a lattice whose dual lattice will have basis elements that define a parallelepiped that is starlike with respect to the origin and contains the intersection of some neighborhood of the origin with a fundamental region of B.

Definition 4. Let B be a finite subgroup of $GL_n(\mathbb{R})$ and fix, F, a fundamental region for B. A lattice Γ^* is *directly associated with* (B, F) if there exists a basis of Γ^*, $\{\gamma_i\}_{i=1}^n$, such that every basis vector lies in the boundary of F and in the intersection of a distinct subset of size $n-1$ of the set of n hyperplanes bounding F; i.e. $\gamma_i \in \bigcap_{\substack{j=1 \\ j \neq i}}^n H_j$ where $\{H_j\}_{j=1}^n$ is the set of hyperplanes bounding F.

Since $\gamma_k \in H_i$ for every $k \neq i$ and $\{\gamma_i\}_{i=1}^n$ is a basis for $\Gamma^* = \hat{\mathbb{Z}}^n c^{-1}$, then $H_i = \text{span}\{\gamma_k : k \neq i\}$. One can check that $P = \{\sum_{i=1}^n t_i \gamma_i : t_i \in [0,1]\}$ is starlike with respect to the origin and for any $0 < \epsilon \leq \min\{|\gamma_i| : i = 1, \ldots, n\}$, $N_\epsilon(\vec{0}) \cap F \subset P$.

In Section 2, we will show that there exists an MRA, MSF, composite dilation Parseval frame wavelet for every finite group. Specifically, finite Coxeter groups, which are generated by reflections through the hyperplanes bounding a fundamental region, are well understood and have been studied as components of affine systems. For example, see the works of Geronimo, Hardin, Kessler, and Mossapust including [14], [20], and [23]. For more information on Coxeter groups see [8], [9], and [15].

2 MSF, Composite Dilation Parseval Frame Wavelets

The goal of this section is to show that with any finite group B (realized as a subgroup of $GL_n(\mathbb{R})$) and any full rank lattice Γ, we can produce MSF composite dilation Parseval frame wavelets. To produce these composite dilation systems we must find appropriate sets to support the Fourier transform of the scaling function, the *scaling set*, and the wavelets, the *wavelet sets*. The desired properties of the scaling and wavelet sets are provided in the following Parseval frame admissibility conditions.

Definition 5. Let B be a finite subgroup of $GL_n(\mathbb{R})$. A lattice $\Gamma = c\mathbb{Z}^n$ $(c \in GL_n(\mathbb{R}))$ is *PF-B-admissible* if there exists a measurable set $R \subset \hat{\mathbb{R}}^n$ such that:
(i) $R \subset W$ where W is a Γ^*-tiling set for $\hat{\mathbb{R}}^n$ and
(ii) R is a B-tiling set for a starlike neighborhood of the origin, S.

Definition 6. Let Γ be PF-B-admissible and let S be a starlike neighborhood of the origin satisfying Definition 5 (ii). A matrix $a \in GL_n(\mathbb{R})$ is *PF-(B,Γ)-admissible* if a is expanding with $S \subset Sa$ and there exist disjoint, measurable sets $R_1, \ldots, R_L \subset Sa \backslash S$ such that:
(i) for all $l = 1 \ldots L$, $R_l \subset W_l$ where W_l is a Γ^*-tiling set for $\hat{\mathbb{R}}^n$ and
(ii) $\bigcup_{l=1}^{L} R_l$ is a B-tiling set for $Sa \backslash S$.

Now we establish that given a, B, Γ satisfying the Parseval frame admissibility conditions, we are able to generate an MRA, MSF, composite dilation Parseval frame wavelet for $L^2(\mathbb{R}^n)$. First, we state a few elementary lemmas that will be useful. For the remaining discussion, we simplify notation by defining

$$e_k(\xi) = |\det(c)|^{\frac{1}{2}} M_k \chi_R(\xi) = |\det(c)|^{\frac{1}{2}} e^{-2\pi i \xi k} \chi_R(\xi) \tag{2}$$

where $c \in GL_n(\mathbb{R})$ and R are obvious from context.

Lemma 1. *Let $\Gamma = c\mathbb{Z}^n$. If W is a Γ^*-tiling set of $\hat{\mathbb{R}}^n$ and $R \subset W$, then $\{e_k : k \in \Gamma\}$ is a Parseval frame for $L^2(R)$. If R is also a B-tiling set for S, then $\left\{\hat{D}_b e_k : b \in B, k \in \Gamma\right\}$ is a Parseval frame for $L^2(S)$.*

Lemma 2. *If, for each $l = 1, \ldots, L$, $\left\{\phi_i^l\right\}_{i=1}^{\infty}$ is a Parseval frame for $L^2(R_l)$ and $R_{l_1} \cap R_{l_2} = \emptyset$ for $l_1 \neq l_2$, then $\bigcup_{l=1}^{L} \left\{\phi_i^l\right\}_{i=1}^{\infty}$ is a Parseval frame for $L^2(\bigcup_{l=1}^{L} R_l)$.*

The following theorem uses the preceding lemmas and the standard MRA arguments.

Theorem 3. *Suppose $\Gamma = c\mathbb{Z}^n$ is PF-B-admissible, $a \in GL_n(\mathbb{R})$ is PF-(B,Γ)-admissible, and let $\psi^l = |\det(c)|^{\frac{1}{2}} \check{\chi}_{R_l}$ for $R_1, \ldots, R_L$ satisfying Definition 6. Then $\Psi = (\psi^1, \ldots, \psi^L)$ is an MRA, MSF, composite dilation Parseval frame wavelet.*

Proof. An MRA, Parseval frame wavelet has a scaling function φ and wavelets ψ^l for $l = 1, \ldots, L$ that generate Parseval frames for the spaces V_0 and W_0 respectively. Let R and S be sets satisfying Definition 5. Define $V_0 = \check{L}^2(S)$ and $V_j = D_a^{-j} V_0 = \check{L}^2(Sa^j)$. Since $S \subset Sa$, then $\{V_j\}_{j=-\infty}^{\infty}$ is a nested sequence with $V_j \subset V_{j+1}$.

Define $\varphi(x) = |\det(c)|^{\frac{1}{2}} \check{\chi}_R(x) = \check{e_0}(\xi)$. Then $[T_k\varphi]\hat{}(\xi) = e_k(\xi)$. Since R is a B-tiling set for S, Lemma 1 tells us that $\left\{\hat{D}_b [T_k\varphi]\hat{} : b \in B, k \in \Gamma\right\}$ is a Parseval frame for $L^2(S)$. Thus, $\{D_b T_k \varphi : b \in B, k \in \Gamma\}$ is a Parseval frame for V_0.

Suppose $L = 1$ so $Sa \backslash S = \bigcup_{b \in B} R_1 b$ and R_1 is contained in a Γ^*-tiling set for $\hat{\mathbb{R}}^n$. Then, with $\psi = |\det(c)|^{\frac{1}{2}} \check{\chi}_{R_1}$, a similar argument provides that $\{D_b T_k \psi : b \in B, k \in \Gamma\}$ is a Parseval from for $L^2(Sa \backslash S)$. Define $W_0 = \check{L}^2(Sa \backslash S)$. Then $W_0 = V_1 \cap V_0^{\perp}$.

The argument demonstrating that this is an MRA follows the standard MRA arguments (for example see [19] or [21]). Since φ generates a Parseval frame for V_0 and ψ generates a Parseval frame for W_0, $\left\{D_a^j D_b T_k \psi : j \in \mathbb{Z}, b \in B, k \in \Gamma\right\}$ is a Parseval frame for $L^2(\mathbb{R}^n)$ and therefore ψ is an MRA, MSF, composite dilation Parseval frame wavelet.

Suppose $L > 1$. Since $R_1, \ldots, R_L$ satisfy Definition 6, then with $\psi^l = |\det(c)|^{\frac{1}{2}} \check{\chi}_{R_l}$, Lemma 2 provides that $\{D_b T_k \psi^l : b \in B, k \in \Gamma, l = 1, \ldots, L\}$ is a Parseval frame for $W_0 = \check{L}^2(Sa \backslash S)$. Hence, $\left\{D_a^j D_b T_k \psi^l : j \in \mathbb{Z}, b \in B, k \in \Gamma, l = 1, \ldots, L\right\}$ is a Parseval frame for $L^2(\mathbb{R}^n)$. Therefore, $\Psi = (\psi^1, \ldots, \psi^L)$ is an MRA, MSF, composite dilation Parseval frame wavelet for $L^2(\mathbb{R}^n)$. □

Next we establish that if B is any finite subgroup of $GL_n(\mathbb{R})$, then every full rank lattice is PF-B-admissible. We first recall that every finite group has a convex fundamental region bounded by hyperplanes through the origin.

Lemma 4. *Every finite subgroup of $GL_n(\mathbb{R})$ has a convex fundamental region bounded by hyperplanes through the origin.*

Let B be a finite subgroup of $GL_n(\mathbb{R})$. Then B is conjugate to a subgroup of the orthogonal group $O(n)$. Let $g \in GL_n(\mathbb{R})$ and $K \subset O(n)$ such that $B = gKg^{-1}$. K has a convex fundamental region, F', bounded by hyperplanes through the origin (see Chapter 3 in [15]). Then $F = g^{-1}F'$ is a convex fundamental region for B. Since the linear transformation g^{-1} takes hyperplanes to hyperplanes and fixes the origin, F is bounded by hyperplanes through the origin. The convexity of the fundamental region is assumed for the remainder of the discussion.

Theorem 5. *If B is a finite group then every lattice $\Gamma = c\mathbb{Z}^n$ is PF-B-admissible.*

Proof. Suppose B is a finite group and $\Gamma = c\mathbb{Z}^n$. Let F be a fundamental region for B bounded by hyperplanes through the origin $(\vec{0})$. Let $\{\gamma_i\}_{i=1}^n$ be a basis for Γ^*. Define $P = \{\sum_{i=1}^n t_i\gamma_i : t_i \in [0,1]\}$. Then P, or any translate of P, is a Γ^*-tiling set for $\hat{\mathbb{R}}^n$. Choose $\alpha \in \hat{\mathbb{R}}^n$ such that $\vec{0}$ is in the interior of $P - \alpha$. Define $R = (P - \alpha) \cap F$. Then $R \subset P - \alpha$. Also, since $P - \alpha$ and F are both starlike with respect to $\vec{0}$, then R is starlike with respect to $\vec{0}$. Thus, $S = \bigcup_{b \in B} Rb$ is a starlike neighborhood of the origin. Therefore, Γ is PF-B-admissible. □

Example 2. The left portion of Figure 2 is magnified to depict how R is constructed and the right portion shows how S is formed by the action of B on R. For this example in $\hat{\mathbb{R}}^2$, $\alpha = (\frac{1}{4}\gamma_1, \frac{1}{16}\gamma_2)$, $B = D_4$, and $\Gamma = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} \mathbb{Z}^2$.

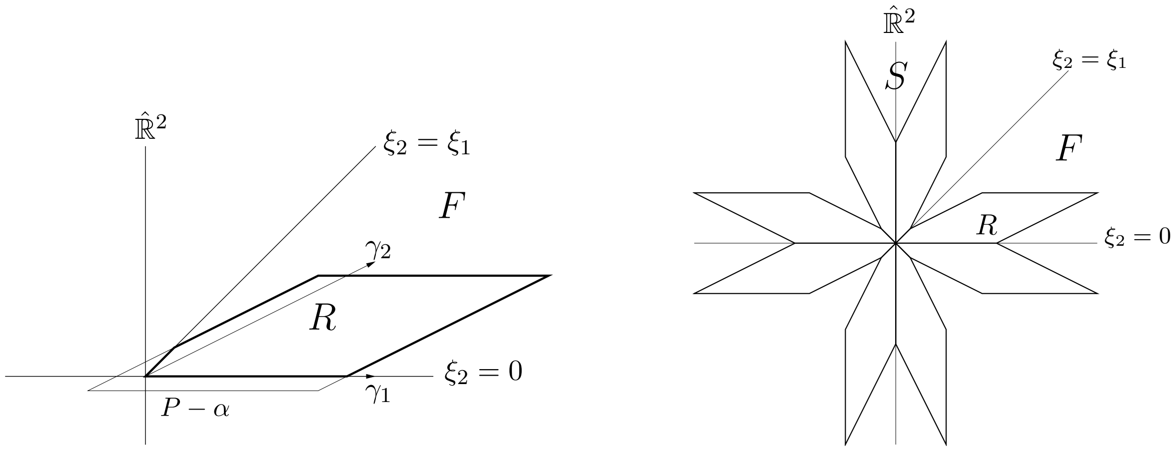

Figure 2: PF-B-admissible: $R = (P - \alpha) \cap F$ and R is a B-tiling set for S.

Finally, we determine sufficient conditions on expanding matrices to satisfy Definition 6. Every expanding matrix, $a \in GL_n(\mathbb{R})$, that normalizes B and satisfies $S \subset Sa$ is PF-(B, Γ)-admissible.

Theorem 6. *Let B be a finite group. Let $\Gamma = c\mathbb{Z}^n$ be a PF-B-admissible lattice with S being a starlike neighborhood of the origin satisfying (ii) from Definition 5. If $a \in GL_n(\mathbb{R})$ is expanding such that $S \subset Sa$ and $aB = Ba$, then a is PF-(B, Γ)-admissible.*

Proof. Suppose R is any set satisfying the conditions of Definition 5 so $S = \bigcup_{b \in B} Rb$. Let $a \in GL_n(\mathbb{R})$ be an expanding matrix such that $S \subset Sa$ and $aB = Ba$. Define $Q = Ra \backslash S$. Then

$$\begin{aligned} \bigcup_{b \in B} Qb &= \bigcup_{b \in B} (Ra \backslash S)\, b = \left\{ \bigcup_{b \in B} Rab \right\} \backslash \left\{ \bigcup_{b \in B} Sb \right\} \\ &= \left\{ \bigcup_{\tilde{b} \in B} R\tilde{b}a \right\} \backslash S = \left(\bigcup_{b \in B} Rb \right) a \backslash S = Sa \backslash S. \end{aligned} \tag{3}$$

Therefore Q is a B-tiling set of $Sa \backslash S$.

Let P be the parallelepiped defined by the lattice basis for Γ^*; then P is a Γ^*-tiling set of $\hat{\mathbb{R}}^n$. Since Q is compact, there exist a smallest integer L and $\eta_1, \ldots, \eta_L \in \Gamma^*$ such that $Q \subset \bigcup_{l=1}^{L} (P + \eta_l)$. Define $R_l = Q \cap (P + \eta_l)$. Then R_l is contained in a Γ^*-tiling set for $\hat{\mathbb{R}}^n$ and by (3), $Q = \bigcup_{l=1}^{L} R_l$ is a B-tiling set of $Sa \backslash S$.

Therefore, a is PF-(B, Γ)-admissible. □

Example 3. As in Example 2, let $\alpha = (\frac{1}{4}\gamma_1, \frac{1}{16}\gamma_2)$, $B = D_4$, and $\Gamma = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} \mathbb{Z}^2$. Then for $a = \frac{3}{2}(I_2)$, Figure 3 shows Sa and that R and Q are B-tiling sets for S and $Sa \backslash S$, respectively.

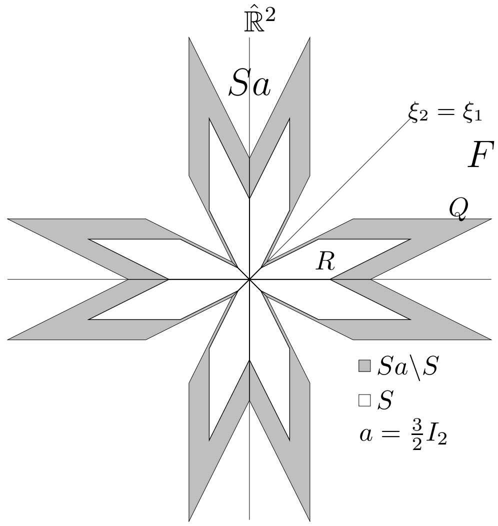

Figure 3: PF-(B, Γ)-admissible: R is B-tiling set for S, Q is a B-tiling set for $Sa \backslash S$.

3 Parseval Frame Wavelets Associated with Orthonormal MSF Composite Dilation Wavelets

In [3], we developed MRA, MSF composite dilation wavelets generating an orthonormal basis for $L^2(\mathbb{R}^n)$. That construction requires $2^n - 1$ wavelet generators. If one is willing to accept a Parseval frame rather than an orthonormal basis, one may easily reduce to a single wavelet generator while maintaining the overall geometry in the Fourier domain. The construction of orthonormal bases has a very minor restriction on the full rank lattices Γ, namely Γ^* must have a basis element pointing to the interior of the chosen fundamental region for B.

We recall the basic construction from [3]. Let B be a finite group and F a fundamental region for B bounded by n hyperplanes through the origin in $\hat{\mathbb{R}}^n$. Choose a full rank lattice, Γ, where Γ^* has a basis ordered so that γ_n points to the interior of F. Construct the parallelepiped formed by the basis vectors; call it $\tilde{P}$. Choose some $\alpha \in \hat{\mathbb{R}}^n$ such that $P = \tilde{P} - \alpha$ contains an open neighborhood of the origin. Take the union of a all translates of P by γ_n and intersect it with $F \backslash F + \gamma_n$, i.e let $R = \left[\bigcup_{j \in \mathbb{Z}} (P + j\gamma_n)\right] \cap [F \backslash F + \gamma_n]$. Then R is a Γ^*-tiling set for $\hat{\mathbb{R}}^n$ and a B-tiling set for a starlike neighborhood of the origin, S. The wavelets are produced by taking our expanding matrix to be $a = 2I_n$. Applying a to R provides $2^n - 1$ sets which are each Γ^*-tiling sets for $\hat{\mathbb{R}}^n$. What is important here is that the union of these sets is a B-tiling set for $Sa \backslash S$.

Example 4. Like Examples 2 and 3, $B = D_4$ and $\Gamma = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} \mathbb{Z}^2$. However, in this example $\alpha = (\frac{1}{2}\gamma_1, \frac{1}{2}\gamma_2)$. Figure 4 depicts the set R and how it is constructed by P. On the right, the figure shows the resulting set $S = \bigcup_{b \in B} Rb$.

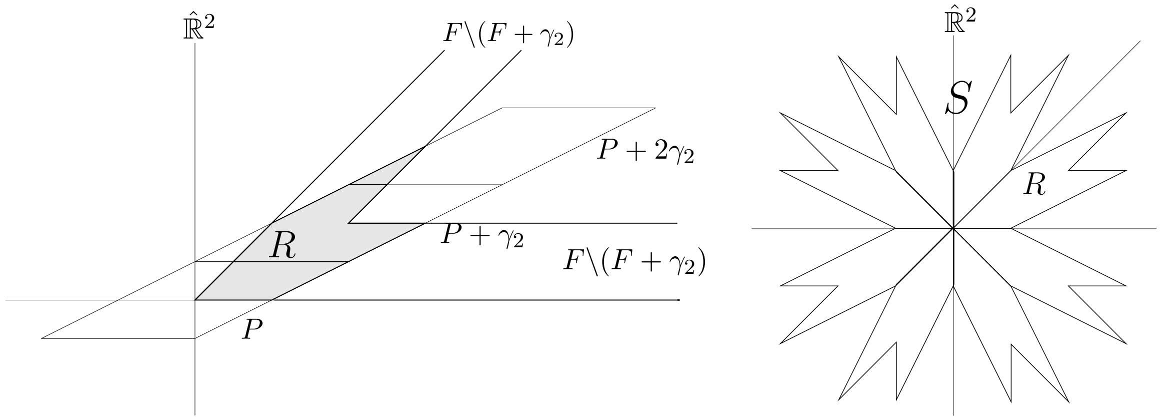

Figure 4: R is a Γ^*-tiling set for $\hat{\mathbb{R}}^n$ and a B-tiling set for S.

In Figure 5, with $a = 2I_2$, the scaling function for our MRA, φ, is defined so that $\hat{\varphi} = |\det(c)|^{\frac{1}{2}} \chi_R$ and each wavelet ψ^1, ψ^2, ψ^3 is defined so that $\hat{\psi}^l = |\det(c)|^{\frac{1}{2}} \chi_{R_l^+ \cup R_l^-}$.

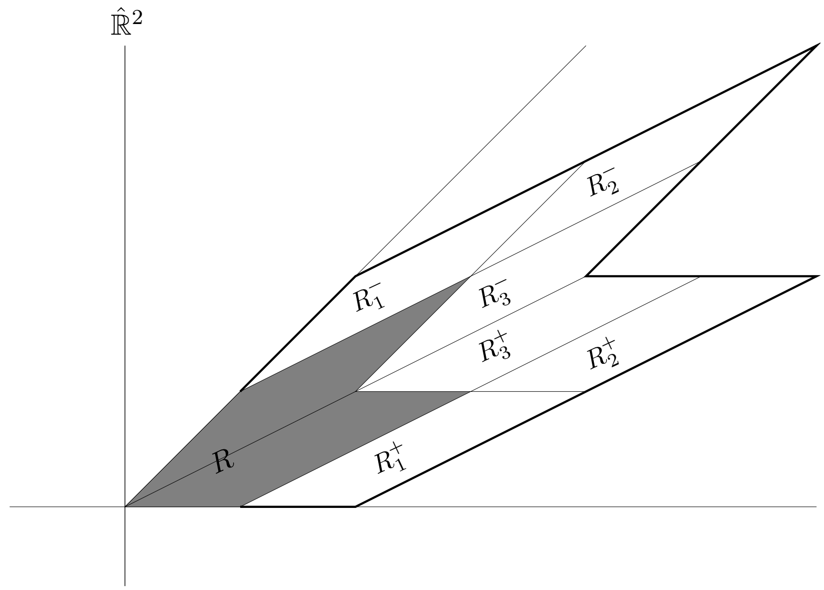

Figure 5: $R\,(2I_n) = R \cup \left[\bigcup_{l=1}^{3} (R_l^+ \cup R_l^-)\right]$.

Theorem 7. *Suppose B is a finite subgroup of $GL_n(\mathbb{R})$ and $\Gamma = c\mathbb{Z}^n$ is such that Γ^* has at least one basis element in the interior of a fundamental region for B. Then there exists a singly generated, MRA, MSF composite dilation Parseval frame wavelet for $L^2(\mathbb{R}^n)$.*

Proof. From Theorem 10 in [3], with $a = 2I_n$, there exists an MRA, MSF, composite dilation wavelet for $L^2(\mathbb{R}^n)$. The preceding discussion reviewed the construction of the appropriate supports sets. Let R be the scaling set and $R_1, \ldots, R_L$ be the wavelet sets for this orthonormal system. Then R is a Γ^*-tiling set $\hat{\mathbb{R}}^n$ and a B-tiling set for a starlike neighborhood, S:

$$\bigcup_{b \in B} Rb = S. \tag{4}$$

Furthermore, $\bigcup_{l=1}^{L} R_l$ is a B-tiling set for $Sa \backslash S$:

$$\bigcup_{b \in B} \left(\bigcup_{l=1}^{L} R_l \right) b = Sa \backslash S. \tag{5}$$

Since $a = 2I_n$, we know from the construction of the support sets that $R \cup \left[\bigcup_{l=1}^{L} R_l \right] = Ra$. Therefore $Ra^{-1} \cup \left[\left(\bigcup_{l=1}^{L} R_l \right) a^{-1} \right] = R$. Define $Q = Ra^{-1}$ and $Q_1 = \left(\bigcup_{l=1}^{L} R_l \right) a^{-1}$. Then $Q, Q_1 \subset R$, a Γ^*-tiling set for $\hat{\mathbb{R}}^n$.

Define $T = Sa^{-1}$. Then T is obviously a starlike neighborhood of the origin and $Ta \backslash T = S \backslash Sa^{-1}$. Applying a^{-1} to (4) verifies that Q is a B-tiling set for T. Therefore, Γ is PF-B-admissible.

Applying a^{-1} to (5) provides that Q_1 is a B-tiling set for $Ta \backslash T$. Since a is expanding and $T \subset Ta$, a is PF-(B, Γ)-admissible. By Theorem 3, with $\psi = |\det(c)|^{\frac{1}{2}} \check{\chi}_{Q_1}$, $\mathcal{A}_{AB\Gamma}(\psi) = \left\{ D_a^j D_b T_k \psi : j \in \mathbb{Z}, b \in B, k \in \Gamma \right\}$ is a Parseval frame for $L^2(\mathbb{R}^n)$. Therefore, ψ is an MRA, MSF composite dilation Parseval frame wavelet for $L^2(\mathbb{R}^n)$. $\square$

Also in [3], we constructed singly generated non-MRA, MSF, composite dilation wavelets for $L^2(\mathbb{R}^n)$. Again, there is a nice advantage to using a Parseval frame rather than an orthonormal basis. In this case, choosing a Parseval frame allows us to retain the MRA structure of our composite dilation system. These constructions require us to choose our lattice so that Γ^* is directly associated to (B, F) for some specified fundamental region F.

Proposition 8. *Suppose B is a finite group with fundamental region, F, bounded by n hyperplanes through the origin. If $\Gamma = c\mathbb{Z}^n$ has a dual lattice Γ^* directly associated with (B, F), then there exists a singly generated, MRA, MSF composite dilation Parseval frame wavelet for $L^2(\mathbb{R}^n)$.*

Proof. Let B be a group with fundamental region, F, bounded by n hyperplanes through the origin. Choose a lattice $\Gamma = c\mathbb{Z}^n$ such that Γ^* is directly associated with (B, F). We let $P = \{\sum_{i=1}^{n} t_i \gamma_i : t_i \in [0, 1]\}$ where $\{\gamma_i\}_{i=1}^{n}$ are the basis vectors of Γ^*. Then P is clearly a Γ^*-tiling set for $\hat{\mathbb{R}}^n$. Let $a = 2I_n$ and define $R = Pa^{-1} = \left\{ \sum_{i=1}^{n} s_i \gamma_i : s_i \in \left[0, \frac{1}{2}\right] \right\}$. Then $R \subset P$. By definition, R is a starlike with respect to the origin. Define $S = \bigcup_{b \in B} Rb$. Then R is a B-tiling set for S and S is a starlike neighborhood of the origin. Hence, with R and S as defined, Γ is PF-B-admissible.

The matrix $a = 2I_n$ is expanding, $S \subset Sa$, and $aB = Ba$. By Theorem 6, a is PF-(B, Γ)-admissible. For clarity, we describe the sets explicitly. Since $R = Pa^{-1}$, then $Ra = P$. Let

$Q = Ra \backslash S = Ra \backslash R$ as in the proof of Theorem 6. Then $R_1 = Ra \backslash R = Q \subset P$. So the scaling function is supported on R and the wavelet on R_1.

With $\psi = |\det(c)|^{\frac{1}{2}} \check{\chi}_{R_1}$, we have $\mathcal{A}_{AB\Gamma}(\psi) = \left\{ D_a^j D_b T_k \psi : j \in \mathbb{Z}, b \in B, k \in \Gamma \right\}$ is a Parseval frame for $L^2(\mathbb{R}^n)$. Therefore, ψ is an MRA, MSF composite dilation Parseval frame wavelet for $L^2(\mathbb{R}^n)$. □

Example 5. Let $B = D_4$ and $a = 2(I_2)$. Let $\Gamma = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mathbb{Z}^2$ so that it is directly associated with (B, F). Then taking P to be the parallelogram formed by the two basis vectors for Γ^*, we construct $R = Pa^{-1}$ as in the proof above. We let $R_1 = Ra \backslash R$. Then the scaling function, φ, is defined by $\hat{\varphi} = \chi_R$ and the wavelet, ψ, is defined by $\hat{\psi} = \chi_{R_1}$.

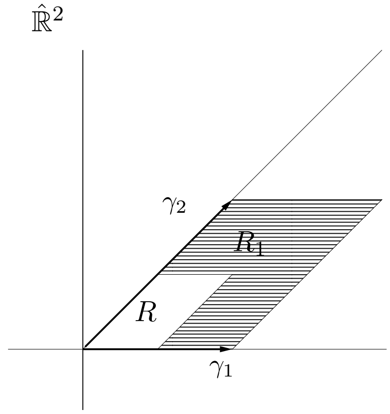

Figure 6: A singly generated, MRA, MSF, composite dilation Parseval frame wavelet.

4 Examples

In this section we present three additional examples to illustrate the versatility of this theory. As noted in the introduction, there has been significant research on finding wavelet-like systems that can capture directional information as well as time and frequency information (for example [5], [6], [13], [16], [24]). Using the action of the composite dilation group B we can obtain essentially arbitrary orientations. With an arbitrary full rank lattice and a well chosen fundamental region for B, we can construct our support sets to approximate the geometry of the Fourier transform of a given class of functions.

Example 6. Suppose we want to capture directional information for multiple orientations in $\hat{\mathbb{R}}^n$. Fix a specified hyperplane, H_1, perpendicular to one of the standard axes. Then choose your composite dilation group B so that the hyperplanes bounding the fundamental region, F, form the desired angle with H_1. When B is generated by reflections through these hyperplanes, each angle must be of the form $\frac{\pi}{m}$ for some $m \in \mathbb{N}$. In this case, when you project onto any hyperplane perpendicular to the two hyperplanes forming the angle $\frac{\pi}{m}$, you obtain orientations in m distinct directions. Let $\frac{\pi}{m_{ij}}$ denote the angle between any two bounding hyperplanes, H_i and H_j. For n hyperplanes bounding the fundamental region of your composite dilation group, B, you obtain $\sum_{i=1}^{n-1}\sum_{j>i} m_{ij}$ distinct orientations. We now have the appropriate orientations to capture our directional information. In order to best capture this directional information choose Γ so that the basis vectors of Γ^* will define a parallelepiped that is short in the directions you are willing to neglect and long in the directions you hope to capture.

For a concrete example in dimension two, let B be the group generated by reflections through the lines $\xi_2 = 0$ and $\xi_2 = \left(\tan\frac{\pi}{5}\right)\xi_1$. Choose $\Gamma = \begin{pmatrix} \frac{4}{25} & 0 \\ -\frac{4}{25}\cot\frac{\pi}{5} & 2\csc\frac{\pi}{25} \end{pmatrix}\mathbb{Z}^2$ so that $\Gamma^* = \hat{\mathbb{Z}}^2 \begin{pmatrix} \frac{25}{4} & 0 \\ \frac{1}{2}\cos\frac{\pi}{5} & \frac{1}{2}\sin\frac{\pi}{5} \end{pmatrix}$ is directly associated with (B,F). Then $|\gamma_1| = 6.25$ and $|\gamma_2| = \frac{1}{2}$. Now we have long, narrow windows that can be oriented in 5 distinct directions. Figure 7 depicts one potential set R_1 such that $\hat{\psi} = |\det(c)|^{\frac{1}{2}}\chi_{R_1}$, the Fourier transform of the Parseval frame wavelet for this example.

Figure 8 shows multiple dilations by elements of B and lattice translations of the support set for $\hat{\psi}$. Notice that we can capture directional information in $m = 5$ distinct orientations. The reflected versions of R_1 are considerably off the lattice, so lattice translates of these sets easily overlap each other, providing redundancy.

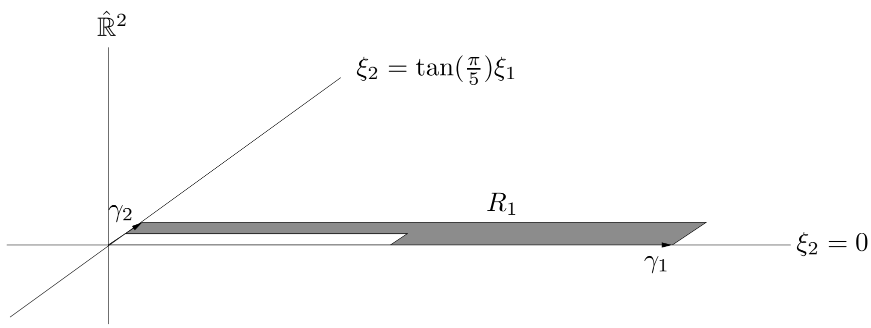

Figure 7: A long, narrow support set for a singly generated, MRA, MSF, composite dilation Parseval frame wavelet.

Example 7. It is possible to construct support sets that are not unions of parallelepipeds. Here we construct sectors of an annulus whose lattice translates provide redundancy.

Let $a = \frac{m}{l}I_n$ for any $l < m \in \mathbb{N}$ and B be a group generated by reflections through n hyperplanes through the origin. Fix F, a fundamental region for B, and let $\Gamma = c\mathbb{Z}^n$ be a lattice such that Γ^* is directly associated with (B,F). Let $t = \min\{|\gamma_i| : i = 1,\ldots,n\}$. Define S to be the n dimensional ball centered at the origin with radius $\frac{l}{m}t$. A ball centered at the origin is invariant under B. Define $R = S \cap F$. Then with R and S, Γ is PF-B-admissible since $R \subset \{\sum_{i=1}^{n} t_i\gamma_i : t_i \in [0,1]\}$ and $S = \bigcup_{b\in B} Rb$ is obviously a starlike neighborhood of the origin.

With $a = \frac{m}{l}I_n$, $R \subset Ra$ and since Ra has radius t, $t \leq |\gamma_i|$ for all $i = 1,\ldots,n$, $Ra \subset \{\sum_{i=1}^{n} t_i\gamma_i : t_i \in [0,1]\}$. Now, with $R_1 = Ra\backslash R$,

$$Sa = \bigcup_{b\in B} Rba = \bigcup_{b\in B} Rab = \left\{\bigcup_{b\in B} R_1 b\right\} \cup \left\{\bigcup_{b\in B} Rb\right\} = \left\{\bigcup_{b\in B} R_1 b\right\} \cup S.$$

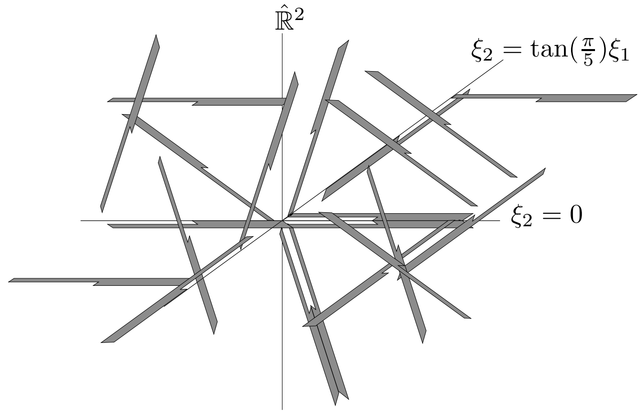

Figure 8: Multiple B-dilations and Γ^*-translations of a long, narrow window for $m = 5$.

Thus $S \subset Sa$, R_1 is contained in a Γ^* tiling set for $\hat{\mathbb{R}}^n$, and R_1 is a B-tiling set for $Sa \backslash S$. Therefore, a is PF-(B, Γ)-admissible. Hence, with $\psi = |\det(c)|^{\frac{1}{2}} \check{\chi}_{R_1}$, we have ψ is an MRA, MSF, composite dilation Parseval frame wavelet for $L^2(\mathbb{R}^n)$.

Figure 9 depicts such wavelets in $\mathbb{R}^2$ when $B = D_4$, $\Gamma = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mathbb{Z}^2$ and, from left to right, $a = \frac{7}{2} I_2$, $a = 2I_2$, and $a = \frac{8}{7} I_2$. In each case the scaling function, φ, is defined by $\hat{\varphi} = \chi_R$ and the wavelet, ψ, is defined by $\hat{\psi} = \chi_{R_1}$.

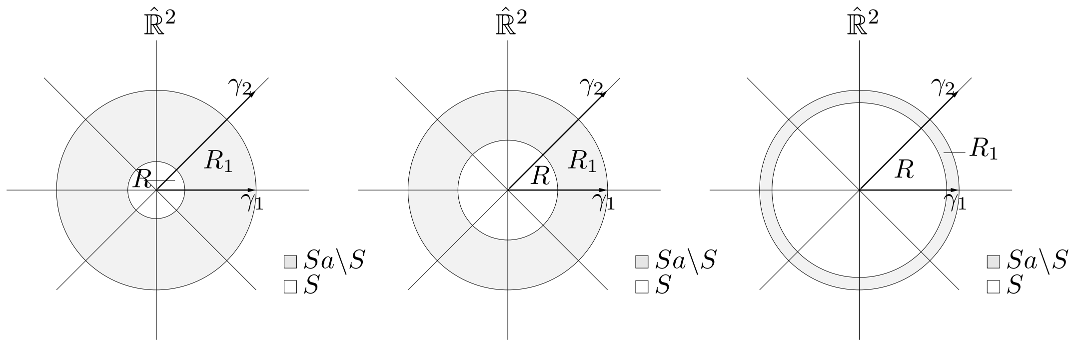

Figure 9: Singly generated, MRA, MSF, composite dilation Parseval frame wavelets from an annulus.

Example 8. Now we present an example due to Alexi Savov [25] which provides a singly generated, MRA, MSF composite dilation Parseval frame wavelet for $L^2(\mathbb{R}^3)$. This system exploits the symmetries of a regular polygon in $\mathbb{R}^2$ by looking at a double pyramid.

Let P be a regular n-gon, $n > 4$, centered at the origin in $\hat{\mathbb{R}}^2$. We will denote $\hat{\mathbb{R}}^2$ as the ξ_1, ξ_2 plane. Let $\tilde{B}$ be the group of symmetries of P and $\tilde{P}$ a $\tilde{B}$-tiling set for P. Figure 10 shows $\tilde{P}$ and P when $n = 5$. When $\tilde{B}$ is the group of symmetries of a pentagon, the triangle $\tilde{P}$ is a $\tilde{B}$-tiling set for the darker pentagon P.

Select a point $h = (0, 0, h)$ on the ξ_3 axis and generate the pyramid, Q, determined by P and h. Define R as the convex hull of $\tilde{P}$ and h. Then $Q = \bigcup_{\tilde{b} \in \tilde{B}} R\tilde{b}$.

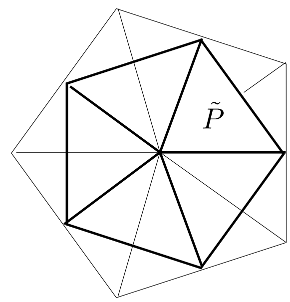

Figure 10: For $n = 5$, $\tilde{P}$ is a $\tilde{B}$-tiling set for P and $P \subset Pa$.

Let r be the reflection through the ξ_1, ξ_2 plane; generate the group B by adding r to $\tilde{B}$. Let $S = Q \cup Qr$. Then B fixes S and $S = \bigcup_{b \in B} Rb$.

Let

$$a = \rho\alpha = \begin{bmatrix} \cos(\frac{\pi}{n}) & \sin(\frac{\pi}{n}) & 0 \\ -\sin(\frac{\pi}{n}) & \cos(\frac{\pi}{n}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\cos(\frac{\pi}{n})} & 0 & 0 \\ 0 & \frac{1}{\cos(\frac{\pi}{n})} & 0 \\ 0 & 0 & 2\cos^2(\frac{\pi}{n}) \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} 1 & \tan(\frac{\pi}{n}) & 0 \\ -\tan(\frac{\pi}{n}) & 1 & 0 \\ 0 & 0 & 2\cos^2(\frac{\pi}{n}) \end{bmatrix}. \tag{7}$$

Since we chose $n > 4$, $\left|\cos(\frac{\pi}{n})\right| > \frac{1}{\sqrt{2}}$. Therefore, the eigenvalues of α are $\left|\frac{1}{\cos(\frac{\pi}{n})}\right| > \sqrt{2}$ and $\left|2\cos^2(\frac{\pi}{n})\right| > 1$. Since ρ is a rotation, $|\det(\rho)| = 1$. Hence, $a = \rho\alpha$ is expanding.

Now we observe that a normalizes B. Since α is diagonal, α will commute with every element of the group B. ρ is a rotation by $-\frac{\pi}{n}$ in the ξ_1, ξ_2 plane and, therefore ρ normalizes $\tilde{B}$. Since r fixes $\hat{\mathbb{R}}^2$, ρ and r commute. Thus, $aBa^{-1} = \alpha\rho B\rho^{-1}\alpha = \rho B\rho^{-1} = B$.

Finally, we show that $S \subset Sa$. First we observe the action of a^{-1} on P, the n-gon base of S in the ξ_1, ξ_2 plane. Let, $v = (v_1, v_2, 0), w = (w_1, w_2, 0)$ be any two adjacent vertices of P. Then $|v| = |w|$, and the angle between v and w is $\frac{2\pi}{n}$. Hence, the midpoint of the edge of P determined by v and w is defined by a vector of length $\cos(\frac{\pi}{n})\,|v|$. Then the inscribed circle of P has radius $\cos(\frac{\pi}{n})\,|v|$.

For all $\xi_0 = (\xi_1, \xi_2, 0) \in P$, we know $|\xi_0| \le |v|$. Since $a^{-1} = \alpha^{-1}\rho^{-1}$ and ρ^{-1} is a rotation of P, then $\left|\xi_0 a^{-1}\right| = \left|\xi_0\alpha^{-1}\right| = \cos(\frac{\pi}{n})\,|\xi_0| \le \cos(\frac{\pi}{n})\,|v|$. Thus, Pa^{-1} is contained in the inscribed circle of P establishing $Pa^{-1} \subset P$. Figure 10 depicts $P \subset Pa$ for a pentagon base.

Now we observe the action of a^{-1} on S. Recall that Q was the pyramid determined by P and a point $(0,0,h)$. Note that $\left|(0,0,h)a^{-1}\right| = \left|(0,0,h)\alpha^{-1}\right| = \frac{1}{2\cos^2(\frac{\pi}{n})}\,|h| < |h|$. Since a^{-1} is also the product of a rotation in the ξ_1, ξ_2 plane and a diagonal matrix, then a^{-1} will take the convex hull of P and $(0,0,h)$ to the convex hull of Pa^{-1} and $(0,0,h)a^{-1}$. With $Pa^{-1} \subset P$ and $\left|(0,0,h)a^{-1}\right| < |h|$, then $Qa^{-1} \subset Q$. Now $Sa^{-1} = [Q \cup Qr]\,a^{-1} = \left\{Qa^{-1}\right\} \cup \left\{Qra^{-1}\right\} = \left\{Qa^{-1}\right\} \cup \left\{Qa^{-1}\right\} r \subset Q \cup Qr = S$. Therefore, $S \subset Sa$.

With our example for $n = 5$, Figure 11 shows the relation $S \subset Sa$ for a defined by equation (7).

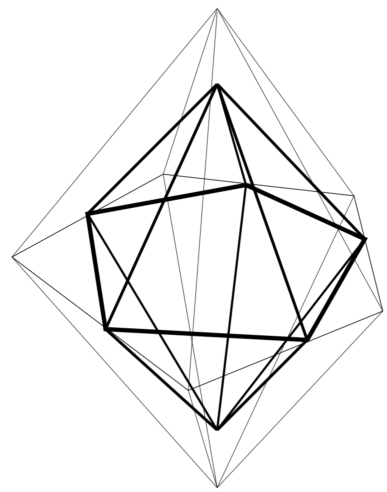

Figure 11: $S \subset Sa$ for a double pyramid with pentagon base.

Now we have a set R that is a B-tiling set of S. We need to choose a lattice $\Gamma = c\mathbb{Z}^n$ such that R is contained in a Γ^*-tiling set of $\hat{\mathbb{R}}^n$. With such a lattice, Γ is PF-B-admissible. Since a is expanding, $S \subset Sa$, and a normalizes B, a is PF-(B, Γ)-admissible. However, to obtain a singly generated composite dilation wavelet, we also need that $R_1 = Ra \backslash S$ is contained in a lattice tiling set of $\hat{\mathbb{R}}^n$. Therefore, we simply choose Γ so that both R and R_1 are contained in lattice tiling sets. Alternatively, we can simply scale S such that R and R_1 are contained in $[0, 1]^3$.

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