Crystallographic Haar-type Composite Dilation Wavelets

Jeffrey D. Blanchard[1] and Kyle R. Steffen[2]

Abstract An (a, B, Γ) composite dilation wavelet system is a collection of functions generating an orthonormal basis for $L^2(\mathbb{R}^n)$ under the actions of translations from a full rank lattice, Γ, dilations by elements of B, a subgroup of the invertible $n \times n$ matrices, and dilations by integer powers of an expanding matrix a. A Haar-type composite dilation wavelet system has generating functions which are linear combinations of characteristic functions. Krishtal, Robinson, Weiss, and Wilson introduced three examples of Haar-type (a, B, Γ) composite dilation wavelet systems for $L^2(\mathbb{R}^2)$ under the assumption that B is a finite group which fixes the lattice Γ. We establish that for any Haar-type (a, B, Γ) composite dilation wavelet, if B fixes Γ, known as the crystallographic condition, B is necessarily a finite group. Under the crystallographic condition, we establish sufficient conditions on (a, B, Γ) for the existence of a Haar-type (a, B, Γ) composite dilation wavelet. An example is constructed in $\mathbb{R}^n$ and the theory is applied to the 17 crystallographic groups acting on $\mathbb{R}^2$ where 11 are shown to admit such Haar-type systems.

Key words: Wavelets, composite dilation wavelets, crystallographic groups, Haar wavelet, plane symmetry groups

1 Introduction

Composite Dilation Wavelets (CDW) [14, 15, 16] are a recent generalization of wavelets [8, 18, among many] obtained from dilation by two countable sets of invertible matrices and translations by elements of a full rank lattice. As with classical wavelets, the theory of composite wavelets has seen early

1. Department of Mathematics and Statistics, Grinnell College, Grinnell, Iowa, USA (jeff@math.grinnell.edu) · 2. Department of Mathematics, University of Utah, Salt Lake City, Utah, USA (kyle.steffen@gmail.com)

advances in the frequency domain and has now moved to the time domain with the introduction of Haar-type composite dilation wavelets [20]. An advantage of the composite dilation setting is the ability to introduce geometric (directional) information into the system as with the orientable oscillatory representation systems: ridglets [5], curvelets [6], shearlets [12], etc. Naturally, it is important to know which composite dilation groups have a collection of functions with which one may construct a composite dilation wavelet system.

In this article, we consider composite dilation wavelets which arise from crystallographic groups and are linear combinations of characteristic functions, which we call crystallographic Haar-type composite dilation wavelets. We provide sufficient conditions under which such Haar-type composite dilation wavelet systems exist. Furthermore, under the crystallographic condition, we establish that the composite dilation group must be finite and provide an example of such a wavelet for an arbitrary dimension. The theory that is developed is applied to the crystallographic groups in $\mathbb{R}^2$ to show that 11 of these 17 groups admit Haar-type composite dilation wavelets.

1.1 Composite Dilation Wavelets

We denote the elements of $\mathbb{R}^n$, considered as column vectors, by letters from the Roman alphabet, usually x, k, or l. We denote elements of the Fourier domain by letters from the Greek alphabet and consider them to be row vectors, usually ξ, γ, or β. The letters f, φ, ψ shall be elements of the set of square integrable functions $L^2(\mathbb{R}^n)$, Φ, Ψ square integrable vector-valued functions, D, T are operators on functions, A, B, G subgroups of $GL_n(\mathbb{R})$ (the group of invertible matrices), and a, b, c matrices, often elements of a subgroup of $GL_n(\mathbb{R})$. Throughout, μ shall denote Lebesgue measure and the term *disjoint* is used in the Lebesgue measure sense. Occasionally, notation may differ but should be clear from context.

For an invertible matrix $c \in GL_n(\mathbb{R})$, define the *dilation of f by c* as $D_c f(x) = |\det c|^{-\frac{1}{2}} f(c^{-1}x)$. Let $\Gamma \subset \mathbb{R}^n$ be a full rank lattice, i.e. there exists $c \in GL_n(\mathbb{R})$ such that $\Gamma = c\mathbb{Z}^n$ and define the *translation of f by $k \in \Gamma$* as $T_k f(x) = f(x-k)$. With these two unitary operators, one constructs an affine system, $\mathcal{A}_{C\Gamma}(\Psi) = \{D_c T_k \psi^\ell : c \in C, k \in \Gamma, 1 \le \ell \le L\}$, from a countable set of invertible matrices, $C \subset GL_n(\mathbb{R})$, a full rank lattice Γ, and a set of generating functions $\Psi = (\psi^1, \dots, \psi^L) \subset L^2(\mathbb{R}^n)$. Introduced by Guo, Labate, Lim, Weiss, and Wilson [14, 15, 16], affine systems with composite dilations are obtained when $C = AB$ is the product of two not necessarily commuting subsets of invertible matrices. In our setting, we further assume that $A = \{a^j : j \in \mathbb{Z}\}$ is a group generated by integer powers of an expanding matrix, a, and B is a subgroup of $GL_n(\mathbb{R})$, namely $\mathcal{A}_{aB\Gamma}(\Psi) = \{D_a^j D_b T_k \psi^\ell : j \in \mathbb{Z}, b \in B, k \in \Gamma, 1 \le \ell \le L\}$.

Definition 1 (Composite Dilation Wavelet). $\Psi = (\psi^1, \ldots, \psi^L) \subset L^2(\mathbb{R}^n)$ is a *composite dilation wavelet* if there exists an expanding matrix a, a group of invertible matrices B, and a full rank lattice Γ such that $\mathcal{A}_{aB\Gamma}(\Psi)$ is an orthonormal basis of $L^2(\mathbb{R}^n)$.

With a trivial composite dilation group, $B = \{I_n\}$ where I_n is the identity on $\mathbb{R}^n$, Def. 1 corresponds to the standard multiwavelet definition. Additionally, if $n = 1$, $\Gamma = \mathbb{Z}$, and $a = 2$, we recover the standard definition of a wavelet. For classical wavelets, a performs expanding dilations of the wavelets while the translations act by "shifting" the supports of the wavelets. The introduction of the composite dilation group B in Def. 1 incorporates more geometric structure to the "shifts" of the wavelet supports. For CDW, a continues to perform the expanding dilations while dilations from B and translations from Γ act together as "shifts."

In wavelet theory, an important property is the notion of a multiresolution analysis. Guo et al. showed that the MRA structure extends naturally the setting of composite dilation wavelets [16].

Definition 2 (MRA). A nested sequence, $\{V_j\}_{j\in\mathbb{Z}}$, of closed subspaces of $L^2(\mathbb{R}^n)$ is an (a, B, Γ)-*multiresolution analysis (MRA)* if all of the following conditions are satisfied:

(M1) $V_j \subset V_{j+1}$ where $V_j = D_a^{-j} V_0$;
(M2) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$;
(M3) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
(M4) there exists $\varphi \in V_0$ such that $\{D_b T_k \varphi : b \in B, k \in \Gamma\}$ is an orthonormal basis for V_0.

A function φ which validates (M4) is called a *composite scaling function* in analogy to a scaling function associated with a classical MRA. Notice that the classical MRA is captured by this definition since a classical wavelet can be thought of as a composite dilation wavelet when B is the trivial group consisting only of the identity. In this case, we have that φ is a scaling function for the MRA if $\{T_k \varphi : k \in \Gamma\}$ is an orthonormal basis for V_0.

Definition 1 is not the most general form of a composite dilation wavelet [14] but is appropriate for the current discussion. Some standard relaxations of the definition replace the requirement for an orthonormal basis with a Riesz basis or Parseval frame. Since the introduction of composite dilation wavelets, significant progress has been made in the theory of shearlets [9, 12, 13, 21, for example] which are smooth functions generating a Parseval frame for $L^2(\mathbb{R}^2)$ and have compact support in frequency. The composite dilation group used to generate the shearlet systems is the group of integer shear matrices:

$$B = \left\{ \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} : j \in \mathbb{Z} \right\}. \tag{1}$$

There is one example of an orthonormal CDW using the the shear group (1) [16]. This is an example of a *minimally supported frequency* (MSF) composite dilation wavelet which consists of characteristic functions of compact sets in the frequency plane [16]. Many examples of MSF composite dilation wavelets were presented in the early composite dilation papers [14, 16], and it is known that every finite group B admits an MRA, MSF composite dilation wavelet [2, 3]. Here, we study which groups B admit an MRA CDW where the wavelets are characteristic functions in the time domain.

1.2 Haar-type Composite Dilation Wavelets

In 1909, Alfred Haar [17] introduced an orthogonal system of functions for analyzing signals which is widely recognized as the first wavelet system. Haar's orthogonal function system neatly captures the information of a signal by storing averages and differences. The Haar system consists of two functions, one capturing averages and the other differences:

$$\varphi(x) = \chi_{[0,1]}(x), \tag{2}$$
$$\psi(x) = \chi_{[0,1/2)}(x) - \chi_{[1/2,1]}(x). \tag{3}$$

In the modern language of wavelets, φ is the Haar scaling function and ψ is the Haar wavelet of an MRA for $L^2(\mathbb{R})$. Clearly, these functions have excellent localization in the time domain. However, since these functions are not continuous, the frequency localization suffers.

The MSF composite dilation wavelets mentioned in Sec. 1.1 provide excellent frequency localization, but the desire to obtain significant time localization remains. Krishtal, Robinson, Weiss, and Wilson [20] present three examples of Haar-type composite dilation wavelets. A composite dilation wavelet is *Haar-type* if it carries two distinguishing features of the Haar wavelet, namely a multiresolution analysis structure and generating functions ψ^ℓ defined by linear combinations of characteristic functions.

A *Haar-type composite dilation wavelet* has an associated MRA and therefore an associated scaling function. Like the Haar wavelet, a Haar-type composite scaling function φ is a normalized characteristic function of a measurable set $R \subset \mathbb{R}^n$, $\varphi(x) = \frac{1}{\sqrt{\mu(R)}}\chi_R(x)$ where μ denotes Lebesgue measure. The Haar-type composite dilation wavelet generators are each a linear combination of a collection of measurable sets $R_{\ell,i} \subset \mathbb{R}^n$, i.e. there exist scalars $\alpha_{\ell,i} \in \mathbb{R}$ such that $\psi^\ell = \sum_i \alpha_{\ell,i}\chi_{R_{\ell,i}}$ for each $1 \le \ell \le L$. In this paper, we first study Haar-type composite dilation scaling functions and then obtain the wavelets from the unitary extension principle [4, 24].

1.3 The Crystallographic Condition

As mentioned above, the first examples of Haar-type composite wavelets were announced by Krishtal, Robinson, Weiss, and Wilson [20] under two assumptions: (i) B is a finite group and (ii) the lattice Γ is invariant under the action of B, i.e. $B(\Gamma) = \Gamma$. In Sec. 2, we show that assumption (ii) implies assumption (i), i.e. that B must be a finite group in a Haar-type composite dilation wavelet system. Assumption (ii) is known as the crystallographic condition in group theory.

Definition 3 (Crystallographic Condition). A group of invertible matrices G and a full rank lattice Γ satisfy the *crystallographic condition* if Γ is invariant under the action of G, i.e. $G(\Gamma) = \Gamma$.

The crystallographic condition is usually applied only to subgroups of the orthogonal group acting on $\mathbb{R}^n$. For CDW, we apply the crystallographic condition more generally. When $B(\Gamma) = \Gamma$, the condition (M4) of Def. 2 can be viewed as requiring that the subspace V_0 is a $(B \ltimes \Gamma)$-invariant space. For $(b_1, k_1), (b_2, d_2) \in B \ltimes \Gamma$, the group operation of the semi-direct product $B \ltimes \Gamma$ is given by

$$(b_1, k_1) \cdot (b_2, k_2) = (b_1 b_2, b_2^{-1} k_1 + k_2).$$

For more on this interpretation in CDW, see [16, 20].

However, the notion of a crystallographic group is used in the standard manner throughout our discussion.

Definition 4. Let G be a group of orthogonal transformations and Γ a full rank lattice for $\mathbb{R}^n$. G is a *crystallographic group* if Γ is a subgroup of G and the quotient group (called the point group) G/Γ is finite.

For example, the shear group (1) and $\mathbb{Z}^2$ satisfy the crystallographic condition (Def. 3) but the semi-direct product of the shear group and $\mathbb{Z}^2$ is not a crystallographic group since the quotient group is infinite. In Sec. 2.1, we prove that assuming the crystallographic condition for Haar-type CDW imposes that $B \ltimes \Gamma$ must be a crystallographic group, i.e that B is finite. This implies, for instance, that no Haar-type CDW exist for the shear group (1).

The crystallographic groups are well-studied as they are the groups of symmetries of $\mathbb{R}^n$. The crystallographic condition derived its name from the important implications of crystallographic groups in the study of crystals. As a result, the crystallographic groups are especially well studied in $\mathbb{R}^3$ (for an introduction see [11]). Also, in $\mathbb{R}^2$, the crystallographic groups are often referred to as wallpaper groups or plane symmetry groups and are commonly introduced in introductory algebra (for example, see [1, 25]). As indicated by the name, these wallpaper groups mathematically describe all repeated, congruent patterns that cover the plane and are often used for their geometric aesthetics.

Our goal is to construct *crystallographic Haar-type composite dilation wavelets* (CHCDW) which are Haar-type CDW where the semi-direct product of the composite dilation group B and the lattice Γ form a crystallographic group, i.e. $B \ltimes \Gamma$ satisfies Def. 4. To do so, we rely on two important measurable sets in $\mathbb{R}^n$, both of which are particular instances of a tiling set.

Definition 5 (Tiling Set). Let G be a group of invertible matrices and let R and W be measurable sets in $\mathbb{R}^n$. Then R is a *G-tiling set* of W if W is a disjoint union of the images of R under the action of G:

$$W = \bigcup_{g \in G} gR \quad \text{with} \quad \mu(g_1 R \cap g_2 R) = 0 \text{ for } g_1 \neq g_2 \in G.$$

When the set W is all of $\mathbb{R}^n$ in Def. 5, we simply refer to R as a *G-tiling set* (with "of $\mathbb{R}^n$" implied). While we utilize the notion of a tiling set in various capacities, there are two distinguished types of tiling sets for a crystallographic group. A set $R \subset \mathbb{R}^n$ is called a *fundamental region* of the crystallographic group $B \ltimes \Gamma$ if R is a $(B \ltimes \Gamma)$-tiling set. The second important subset of $\mathbb{R}^n$ is a *Γ-tiling set* which tiles $\mathbb{R}^n$ under the action of translations from the lattice Γ. In the construction of CHCDW, we will see how both of these sets play an important role. For example, the support set of a composite scaling function will always be a fundamental region for the group $B \ltimes \Gamma$ and therefore satisfy (M4) from Def. 2.

For composite dilation wavelets satisfying Def. 1, a crystallographic group must have a factorization into the semi-direct product $B \ltimes \Gamma$ in order to apply dilations from B and translations from Γ. In this case, since B is a subgroup of $B \ltimes \Gamma$, it is clear that $B(\Gamma) = \Gamma$ (see Lem. 2). This factorization into a semi-direct product is not necessary for producing wavelets in a more general setting. At the time of writing this article, the authors became aware of independent work by MacArthur and Taylor [23] in which a more general representation theoretic argument constructs Haar-type wavelet systems for $L^2(\mathbb{R}^2)$.

2 Crystallographic Composite Dilation Wavelets

Throughout this section, we study the implication of the crystallographic condition on Haar-type composite dilation wavelets. We first demonstrate that imposing the crystallographic condition, $B(\Gamma) = \Gamma$, on a Haar-type composite dilation system in fact forces us to use a group and lattice which form a crystallographic group, i.e. $B \ltimes \Gamma$ satisfies Def. 4. This is followed by establishing sufficient conditions for the existence of an (a, B, Γ)-MRA with a Haar-type composite scaling function. From this MRA and composite scaling function, we explore a method for constructing the CHCDW by first obtaining a low pass filter matrix and subsequently producing high pass filter matrices which define the composite dilation wavelets. For the remainder, we consider only Haar-type composite dilation wavelets. Therefore, $\varphi(x) = \frac{1}{\sqrt{\mu(R)}}\chi_R(x)$ for some measurable set $R \subset \mathbb{R}^n$.

2.1 Implications of the Crystallographic Condition

In this section, we investigate the impact of the crystallographic condition on CHCDW. We proceed through a sequence a lemmas demonstrating some geometric implications of imposing the crystallographic condition on our composite dilation system. The main result of this section, Thm. 1, states that the group B must be a finite subgroup of $\widetilde{SL}_n(\mathbb{R})$, the group of invertible matrices with determinant of absolute value 1, and therefore, $B \ltimes \Gamma$ is a crystallographic group.

Lemma 1. *If* $\{D_b T_k \varphi(x) : b \in B, k \in \Gamma\}$ *is an orthogonal system, then* $\cup_{b\in B} bR$ *is a disjoint union.*

Proof. This is an obvious consequence of orthogonality: Let $b_1, b_2 \in B$. If $b_1 \neq b_2$, then

$$0 = \mu(R)\langle D_{b_1}\varphi(x), D_{b_2}\varphi(x)\rangle = \int_{\mathbb{R}^n} \chi_{b_1R}(x)\chi_{b_2R}(x)\ d\mu = \mu\left(b_1R \cap b_2R\right).$$

□

A straightforward interpretation of Lem. 1 is that R must be a B-tiling set of $S = \cup_{b\in B} bR$ in order for $\varphi = \frac{1}{\sqrt{\mu(R)}}\chi_R$ to generate an orthonormal basis for the scaling space V_0 of the MRA in a Haar-type composite dilation wavelet system.

Lemma 2. *If* $B(\Gamma) \subset \Gamma$ *then* $|\det(b)| = 1$ *for all* $b \in B$ *and* $B(\Gamma) = \Gamma$.

Proof. Since $\Gamma = c\mathbb{Z}^n$ for some $c \in GL_n(\mathbb{R})$, there exists $0 \neq k \in \Gamma$ such that $\|k\|_2 \leq \|l\|_2$ for all $l \in \Gamma\backslash 0$. Let $g \in GL_n(\mathbb{R})$ with $|\det(g)| < 1$. Then $0 < \|gk\|_2 < \|k\|_2$ and therefore $gk \notin \Gamma$ so $g \notin B$. Furthermore, $g^{-1} \notin B$ since B is a group. Finally, since I_n is an element of the group B, then $\Gamma \subset B(\Gamma)$ and, hence, $B(\Gamma) = \Gamma$. □

We now show that the disjoint union of the images of the scaling set R under the action of the group B is necessarily contained in a Γ-tiling set if B and Γ satisfy the crystallographic condition.

Lemma 3. *Suppose* $\{D_b T_k \varphi(x) : b \in B, k \in \Gamma\}$ *is an orthogonal system. If* $B(\Gamma) = \Gamma$*, then* $\cup_{b\in B} bR$ *is contained in a* Γ*-tiling set of* $\mathbb{R}^n$.

Proof. It suffices to show that for any $0 \neq k \in \Gamma$,

$$\mu\left((\cup_{b\in B} bR)\cap(\cup_{b\in B} bR)+k\right)=0. \tag{4}$$

Let $b_1, b_2 \in B$. Then

$$\begin{aligned}\mu\left(b_1R\cap(b_2R+k)\right) &= \int_{\mathbb{R}^n}\chi_{b_1R}(x)\chi_{(b_2R+k)}(x)\ d\mu \\ &= \int_{\mathbb{R}^n}\chi_R(b_1^{-1}x)\chi_R(b_2^{-1}x-b_2^{-1}k)\ d\mu \\ &= \mu(R)\int_{\mathbb{R}^n} D_{b_1}\varphi(x)D_{b_2}T_{b_2^{-1}k}\varphi(x)\ d\mu. \end{aligned} \tag{5}$$

Since $B(\Gamma)=\Gamma$, there exists $l\in\Gamma$ such that $l=b_2^{-1}k\neq 0$. Thus, invoking (5) and the assumption that $\{D_bT_k\varphi(x) : b\in B, k\in\Gamma\}$ is an orthogonal system, we have

$$\mu\left(b_1R\cap(b_2R+k)\right)=\mu(R)\left\langle D_{b_1}\varphi(x), D_{b_2}T_l\varphi(x)\right\rangle=0. \tag{6}$$

Since $b_1, b_2\in B$ and $k\in\Gamma$ were all arbitrary, (4) follows directly from (6). □

The preceding lemmas combine to establish that the crystallographic condition on B and Γ imposes that B must be finite if B and Γ are to generate a Haar-type composite dilation wavelet system.

Theorem 1. *Suppose $\{D_bT_k\varphi(x) : b\in B, k\in\Gamma\}$ is an orthogonal system. If $B(\Gamma)=\Gamma$, then B is a finite subgroup of $\widetilde{SL}_n(\mathbb{R})$ and $B\ltimes\Gamma$ is a crystallographic group.*

Proof. From Lem. 3, there exists a Γ-tiling set, W, such that $\cup_{b\in B} bR\subset W$. Furthermore, since $\Gamma=c\mathbb{Z}^n$ for $c\in GL_n(\mathbb{R})$, it is clear that

$$\mu\left(\cup_{b\in B} bR\right)\leq\mu(W)=|\det(c)|<\infty. \tag{7}$$

Now, Lems. 1 and 2 imply

$$\mu\left(\cup_{b\in B} bR\right)=\sum_{b\in B}|\det(b)|\mu(R)=|B|\mu(R). \tag{8}$$

Since $\mu(R)\neq 0$, it follows that $|B|<\infty$. That B is a subgroup of $\widetilde{SL}_n(\mathbb{R})$ follows from Lem. 2. Since B is finite, $B\ltimes\Gamma$ clearly satisfies Def. 4. □

The proof of Thm. 1 provides a geometric insight into how the crystallographic condition imposes that the group B must be finite. For example, although the shear group (1) fixes $\mathbb{Z}^2$, this is an infinite group and it is impossible to use the shear group to generate a Haar-type CDW. In related work, Houska [19] has proven a stronger result, namely that every composite dilation Bessel system with generating functions which are characteristic functions must have a finite group B. Houska's theorem directly implies that B must be finite in a Haar-type composite dilation wavelet system. While we were aware of this result, the preceding argument provides a direct, geometric connection between the crystallographic condition and the size of the group B, and the proof is distinct from Houska's.

2.2 Sufficient Conditions

We now turn our attention to establishing sufficient conditions on a, B, and Γ for the existence of a Haar-type (a, B, Γ)-composite dilation wavelet system for $L^2(\mathbb{R}^n)$. In Sec. 2.1, we saw that when B and Γ satisfy the crystallographic condition, the union of the images of the scaling set R under the action of B must be contained in a Γ-tiling set. Our first lemma states that if this union is a Γ-tiling set, then the scaling function $\varphi = \frac{1}{\sqrt{\mu(R)}}\chi_R$ generates an orthonormal basis for a $(B \ltimes \Gamma)$-invariant space.

Lemma 4. *Suppose $B(\Gamma) = \Gamma$, R is a B-tiling set of S, and $\varphi = \frac{1}{\sqrt{\mu(R)}}\chi_R$. If $S = \cup_{b\in B} bR$ is a Γ-tiling set of $\mathbb{R}^n$, then $\{D_b T_k \varphi : b \in B, k \in \Gamma\}$ is an orthonormal system.*

Proof. Let $b_1, b_2 \in B$ and $k_1, k_2 \in \Gamma$. Since $B(\Gamma) = \Gamma$, there exist $l_1, l_2 \in \Gamma$ such that $l_1 = b_1 k_1$ and $l_2 = b_2 k_2$. By assumption, R is a B-tiling set of S and $b_1 R, b_2 R \subset S$ with S a Γ-tiling set of $\mathbb{R}^n$. Thus, for any $k \in \Gamma$ and any $b \in B$,

$$\mu(b_1 R + k \cap b_2 R + k) = 0 \quad \text{for} \quad b_1 \neq b_2 \quad \text{and} \tag{9}$$

$$\mu(bR + k_1 \cap bR + k_2) = 0 \quad \text{for} \quad k_1 \neq k_2. \tag{10}$$

Hence

$$\begin{aligned}\mu(R)\,\langle D_{b_1} T_{k_1}\varphi, D_{b_2} T_{k_2}\varphi\rangle &= \int_{\mathbb{R}^n} \chi_R(b_1^{-1}x - k_1)\chi_R(b_2^{-1}x - k_2)\; d\mu \\ &= \int_{\mathbb{R}^n} \chi_{(b_1 R + b_1 k_1)}(x)\chi_{(b_2 R + b_2 k_2)}(x)\; d\mu \\ &= \int_{\mathbb{R}^n} \chi_{(b_1 R + l_1)}(x)\chi_{(b_2 R + l_2)}(x)\; d\mu \\ &= \mu(b_1 R + l_1 \cap b_2 R + l_2). \end{aligned} \tag{11}$$

Now, from Lem. 2, $\mu(bR + k) = |\det(b)|\mu(R) = \mu(R)$ for every $b \in B$ and $k \in \Gamma$. Therefore, combining Equations (9)-(11) yields

$$\langle D_{b_1} T_{k_1}\varphi, D_{b_2} T_{k_2}\varphi\rangle = \delta_{b_1,b_2}\delta_{k_1,k_2}$$

where δ is the usual Kronecker delta function. □

Theorem 2 (Sufficient Conditions for an MRA). *Suppose $B(\Gamma) = \Gamma$. Let $\varphi = \frac{1}{\sqrt{\mu(R)}}\chi_R$ and $V_0 = \overline{span}\{D_b T_k \varphi : b \in B, k \in \Gamma\}$. Let $a \in GL_n(\mathbb{R})$ be an expanding matrix with $|\det(a)| = L + 1$.*

(i) Suppose $S = \cup_{b\in B} bR$ is a disjoint union and a Γ-tiling set of $\mathbb{R}^n$.
(ii) Suppose $a\Gamma \subset \Gamma$ and a normalizes B (i.e. $aB = Ba$).
(iii) Suppose there exist $b_0, \ldots, b_L \in B$ and $k_0, \ldots, k_L \in \Gamma$ such that

$$aR = \bigcup_{i=0}^{L} (b_i R + b_i k_i). \tag{12}$$

Then the sequence of subspaces $\{V_j := D_a^{-j} V_0\}_{j\in\mathbb{Z}}$ is an MRA for $L^2(\mathbb{R}^n)$ and φ is a composite scaling function for this MRA.

Proof. From Lem. 4, condition (i) implies (M4). Since a is expanding, $\lim_{j\to\infty} \mu(a^j R) = \infty$. Therefore the only square integrable function in every closed subspace V_j is the zero function which establishes (M3). As the piecewise constant functions are dense in $L^2(\mathbb{R}^n)$ and V_j is an approximation space spanned by $\{\chi_{a^{-j}(bR+bk)} : j \in \mathbb{Z}, b \in B, k \in \Gamma\}$, the closure of the union of all such approximation spaces is clearly $L^2(\mathbb{R}^n)$, verifying (M2).

Finally, we verify (M1). Since $V_j = D_a^{-j} V_0$, it suffices to show that $D_b T_k \varphi \in V_1$ for any $b \in B$, $k \in \Gamma$ as in the standard MRA arguments. From (12), $R = \cup_{i=0}^{L} a^{-1}(b_i R + b_i k_i)$ which is a disjoint union as argued in the proof Lem. 4. Thus

$$\chi_R(x) = \chi_{(\cup_{i=0}^{L} a^{-1}(b_i R + b_i k_i))}(x) = \sum_{i=0}^{L} \chi_R(b_i^{-1} a x - k_i),$$

and therefore

$$\varphi(x) = |\det(a)|^{-\frac{1}{2}} \sum_{i=0}^{L} D_a^{-1} D_{b_i} T_{k_i} \varphi(x). \tag{13}$$

Now for any $b \in B$, $k \in \Gamma$, (13) implies

$$\begin{aligned} D_b T_k \varphi(x) = \varphi(b^{-1}x - k) &= |\det(a)|^{-\frac{1}{2}} \sum_{i=0}^{L} D_a^{-1} D_{b_i} T_{k_i} \varphi(b^{-1}x - k) \\ &= |\det(a)|^{-\frac{1}{2}} \sum_{i=0}^{L} D_b T_k D_a^{-1} D_{b_i} T_{k_i} \varphi(x). \end{aligned} \tag{14}$$

With $a\Gamma \subset \Gamma$ and $B(\Gamma) = \Gamma$, there exist $l_i = b_i^{-1} a k \in \Gamma$. Hence, $T_k D_a^{-1} D_{b_i} = D_a^{-1} D_{b_i} T_{l_i}$. As a normalizes B, there exists $\tilde{b} \in B$ such that $ab^{-1} = \tilde{b}^{-1} a$ and therefore $D_b D_a^{-1} = D_a^{-1} D_{\tilde{b}}$. With these commuting relationships on the operators we can write (14) as

$$D_b T_k \varphi(x) = |\det(a)|^{-\frac{1}{2}} \sum_{i=0}^{L} D_a^{-1} D_{\tilde{b}b_i} T_{(l_i+k_i)} \varphi(x) \in V_1, \tag{15}$$

which verifies (M1). □

To summarize, to obtain a Haar-type composite scaling function for an (a, B, Γ)-MRA, with $B \ltimes \Gamma$ a crystallographic group, it suffices to find a fundamental region of $B \ltimes \Gamma$ and an expanding matrix $a \in GL_n(\mathbb{R})$ such that $a\Gamma \subset \Gamma$, $aB = Ba$, and the action of a on the fundamental region, R, satisfies condition (iii).

2.3 Low Pass Filter Design

In the development of wavelet theory, properties of filters have played an important role. For example, Daubechies' construction of arbitrarily smooth compactly supported wavelets [7] was accomplished in part by factoring the filter associated with the scaling function. Moreover, the most common implementation algorithms rely completely on the filters. The notion of filters extends to MRA composite dilation wavelets [4, 16] and is useful for constructing wavelets from the composite scaling function.

As outlined in [4, 20], one may view the basis $\{D_b T_k \varphi\}$ of the space V_0 satisfying (M4) of Def. 2 as a Γ-invariant system with $|B|$ generating functions. Define $|B| = m+1$ and fix an ordering to the elements of the composite dilation group, $B = \{b_0 = I_n, b_1, \ldots, b_m\}$. Define the $m+1$ functions

$$\varphi_s(x) := D_{b_s}\varphi(x) = \varphi(b_s^{-1}x) \quad \text{for} \quad 0 \le s \le m. \tag{16}$$

Assuming the crystallographic condition, we have $D_b T_k = T_{bk} D_b$ with $bk \in \Gamma$, and hence, $\{D_b T_k \varphi : b \in B, k \in \Gamma\} = \{T_k D_b \varphi : k \in \Gamma, b \in B\} = \{T_k \varphi_s : k \in \Gamma, 0 \le s \le m\}$. Therefore, we may interpret the composite dilation scaling function as a multi-scaling function with the group structure of B defining each of the scaling functions from the single function φ. Finally, we define the vector valued scaling function $\Phi : \mathbb{R}^n \to \mathbb{R}^{m+1}$ by

$$\Phi(x) = \begin{bmatrix} \varphi_0(x) \\ \vdots \\ \varphi_m(x) \end{bmatrix}. \tag{17}$$

For notational brevity, condition (iii) of Thm. 2 allows elements of the group B to appear in the set $\{b_0, \ldots, b_L\}$ multiple times. Obviously, we could have $m < L$, requiring that for some $0 \le i < i' \le L$, $b_i = b_{i'}$. For the general construction of the low pass filter matrix, we introduce slightly more cumbersome notation by stating the equivalent condition (iii′) Let $I \subset \{0, \ldots, m\}$ be an index set and for each $i \in I$, define the index set P_i. Suppose there exist $\{b_i : i \in I\} \subset B$ and $\{k_{i,p} : p \in P_i\} \subset \Gamma$ with $\sum_{i \in I} |P_i| = L + 1$ such that

$$aR = \bigcup_{i \in I} \bigcup_{p \in P_i} (b_i R + b_i k_{i,p}). \tag{18}$$

Replacing (iii) with (iii′) in Thm. 2 yields the same conclusion as the two conditions are equivalent. To track the commuting relationships, we further define the following index notation. Let $\tilde{s} \in \{0, \ldots, m\}$ be defined by the normalizing relationship $b_{\tilde{s}} = a^{-1} b_s a$ and $s(i) \in \{0, \ldots, m\}$ be defined by the right orbits of the elements of B, namely $b_{s(i)} = b_s b_i$. Since a normalizes the group B, both $\tilde{s}$ and $s(i)$ are well-defined.

With this notation and assumption (iii′), following the proof of Thm. 2 and letting $k = 0$ in (15),

$$\begin{aligned} D_{b_s} \varphi(x) &= |\det(a)|^{-\frac{1}{2}} \sum_{i \in I} \sum_{p \in P_i} D_a^{-1} D_{b_{\tilde{s}} b_i} T_{k_{i,p}} \varphi(x) \\ &= |\det(a)|^{-\frac{1}{2}} \sum_{i \in I} \sum_{p \in P_i} D_a^{-1} D_{b_{\tilde{s}(i)}} T_{k_{i,p}} \varphi(x). \end{aligned} \tag{19}$$

In the multi-scaling function notation (16), the commuting relationship $D_b T_k = T_{bk} D_b$ transforms (19) into

$$\begin{aligned} \varphi_s(x) &= |\det(a)|^{-\frac{1}{2}} \sum_{i \in I} \sum_{p \in P_i} D_a^{-1} T_{\left(b_{\tilde{s}(i)} k_{i,p}\right)} \varphi_{\tilde{s}(i)}(x) \\ &= \sum_{i \in I} \sum_{p \in P_i} T_{\left(b_{\tilde{s}(i)} k_{i,p}\right)} \varphi_{\tilde{s}(i)}(ax). \end{aligned} \tag{20}$$

Taking the Fourier transform, $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi x}\, d\mu$, of (20) results in the equation

$$\hat{\varphi}_s(\xi) = |\det(a)|^{-1} \sum_{i \in I} \sum_{p \in P_i} e^{-2\pi i (\xi a^{-1})(b_{\tilde{s}(i)} k_{i,p})} \hat{\varphi}_{\tilde{s}(i)}(\xi a^{-1}). \tag{21}$$

From (21) we obtain a general construction for the low pass filter matrix.

Corollary 1. *Suppose conditions (i), (ii), and (iii′) for Thm. 2 are satisfied with $I, P, \tilde{s}, s(i)$ defined above and let $|B| = m + 1$. Define the matrix $M_0(\xi)$ of size $(m+1) \times (m+1)$ whose $s, \tilde{s}(i)$ entry is*

$$[M_0(\xi)]_{s,\tilde{s}(i)} := \begin{cases} \frac{1}{|\det(a)|} \sum_{p \in P_i} e^{-2\pi i (\xi a^{-1})(b_{\tilde{s}(i)} k_{i,p})} & \text{for } i \in I, \\ 0 & \text{for } i \notin I. \end{cases} \tag{22}$$

Then $\hat{\Phi}(\xi a) = M_0(\xi) \hat{\Phi}(\xi)$.

Proof. Let $\hat{\Phi}(\xi)$ be the element-wise Fourier transform of $\Phi(x)$ defined by (17). The sth element of $\hat{\Phi}(\xi a)$ is then defined by (21), which is clearly the vector product of the sth row of $M_0(\xi)$ and $\hat{\Phi}(\xi)$ by (22). □

2.4 Haar-type CDW from the Composite Scaling Function

As with the traditional wavelets, the principal task in constructing an MRA wavelet system is obtaining a scaling function. Theorem 2 provides sufficient conditions for the construction of Haar-type composite scaling functions. In this section, we discuss two, closely related, systematic methods for obtaining appropriate composite dilation wavelet generating functions.

In [4], the first author and Krishtal established conditions on matrix filters which can be used to construct crystallographic composite dilation wavelets. This condition is the generalization of the unitary extension principle. We say that $M_0(\xi)$ is a *low pass filter* for Φ if $\hat{\Phi}(\xi) = M_0(\xi a^{-1})\hat{\Phi}(\xi a^{-1})$ and that $M_\ell(\xi)$ is a *high pass filter* for $\Psi^\ell = (\psi_0^\ell, \ldots, \psi_m^\ell)^t$ (defined similarly as (16)) if $\hat{\Psi}^\ell(\xi) = M_\ell(\xi a^{-1})\hat{\Psi}^\ell(\xi a^{-1})$. Define $\hat{\mathbb{Z}}^n$ as the integer lattice of row vectors in the frequency domain, and let $\Gamma^* = \hat{\mathbb{Z}}^n c^{-1}$ denote the dual lattice to $\Gamma = c\mathbb{Z}^n$ so that $\gamma k \in \mathbb{Z}$ for any $\gamma \in \Gamma^*, k \in \Gamma$. The result from [4] is restated here with an appropriate notational modification.

Theorem 3 ([4]). *Let $B = \{b_0 = I, b_1, \ldots, b_m\}$ and $\{V_j\}_{j=-\infty}^{\infty}$ be an MRA for $L^2(\mathbb{R}^n)$ with the associated composite scaling function Φ and the low pass filter $M_0(\xi)$. Let also $\{\beta_0, \ldots, \beta_L\}$ be a full set of coset representatives of $\Gamma^* a^{-1}/\Gamma^*$, $\Psi^\ell = (\psi_0^\ell, \ldots, \psi_m^\ell)^t$ with $\psi_s^\ell \in V_1$ for $s = 0, \ldots, m$, and M_ℓ, $1 \le \ell \le L$, be the associated high pass filters. Then $\Psi = (\psi_s^\ell) \subset L^2(\mathbb{R}^n)$, $0 \le s \le m$, $1 \le \ell \le L$, is a composite dilation wavelet for the (a, B, Γ)-MRA $\{V_j\}_{j=-\infty}^{\infty}$ if and only if*

$$\sum_{i=0}^{L} M_\ell(\xi + \beta_i) M_{\ell'}^*(\xi + \beta_i) = \delta_{\ell,\ell'} I_{m+1} \text{ for } 0 \le \ell, \ell' \le L. \tag{23}$$

With this result, the wavelets may be obtained by constructing appropriate high pass filter matrices from the low pass filter matrix. When $|\det(a)| = 2$, the full set of coset representatives is $\{0, \beta\}$ for some appropriate $\beta \in \hat{\mathbb{R}}^n$. In this case, it was shown in [4] that when $M_0(\xi)M_0^*(\xi) = 0$, $M_1(\xi) = M_0(\xi + \beta)$ is a suitable high pass filter matrix which defines a composite dilation wavelet. A low pass filter for a crystallographic Haar-type composite scaling function (22) always satisfies the requirement $M_0(\xi)M_0^*(\xi) = 0$.

Alternatively, to define the CHCDW in the time domain, we might simply construct $L = |\det(a)| - 1$ functions $\psi^\ell, \ell = 1, \ldots, L$, which are orthogonal to φ yet share the same support as φ. Sharing the same support imposes on these functions the familiar requirement $\int_{\mathbb{R}^n} \psi^\ell(x)\, d\mu = 0$. Since we are after Haar-type functions, we only need to alter the coefficients for the characteristic functions which define the dilated scaling scaling function to construct an orthogonal function. A general approach is to find an orthogonal matrix of size $(L+1)\times(L+1)$ whose first row consists entirely of the constant $|\det(a)|^{-1/2}$. This first row represents the coefficients of the characteristic functions which define the dilated scaling function and ensures that the remaining row sums will all be zero.

Suppose $Q = (q_{\ell,t})_{\ell,t=0}^L$, with $q_{0,t} = 1/\sqrt{L+1} = |\det(a)|^{-1/2}$, is an orthogonal matrix. From (12), the dilated scaling function $\varphi(a^{-1}x)$ is supported on $L+1$ disjoint sets which we relabel $R_t, t = 0, \dots, L$. Then

$$D_a\varphi(x) = |\det(a)|^{-1/2}\varphi(a^{-1}x) = \frac{1}{\sqrt{\mu(R)}}\sum_{t=0}^{L} q_{0,t}\chi_{R_t}(x). \tag{24}$$

Now we define the wavelets by inserting coefficients obtained from the ℓth row of Q.

$$D_a\psi^\ell(x) = |\det(a)|^{-1/2}\psi^\ell(a^{-1}x) = \frac{1}{\sqrt{\mu(R)}}\sum_{t=0}^{L} q_{\ell,t}\chi_{R_t}(x). \tag{25}$$

Then, with $\psi^0 = \varphi$, we have

$$\begin{aligned}
\left\langle \psi^\ell, \psi^{\ell'} \right\rangle &= \int_{\mathbb{R}^n} \psi^\ell(x)\overline{\psi^{\ell'}(x)}\, d\mu \\
&= \frac{|\det(a)|}{\mu(R)} \int_{\mathbb{R}^n} \left[\sum_t q_{\ell,t}q_{\ell',t}\chi_{R_t}(ax) + \sum_{t\neq t'} q_{\ell,t}q_{\ell',t'}\chi_{R_t}(x)\chi_{R_{t'}}(ax) \right] d\mu \\
&= \frac{|\det(a)|}{\mu(R)} \sum_t q_{\ell,t}q_{\ell',t} \int_{\mathbb{R}^n} \chi_{R_t}(ax)\, d\mu \\
&= \frac{|\det(a)|}{\mu(R)} \mu(a^{-1}R) \sum_t q_{\ell,t}q_{\ell',t} \\
&= \delta_{\ell,\ell'}
\end{aligned} \tag{26}$$

with the validity of this sequence of equalities following from (24) and (25), that R_t and $R_{t'}$ are disjoint for $t \neq t'$, and that Q is an orthogonal matrix.

Since each of the functions $\varphi_s(x)$ have disjoint supports and the functions $\psi_s^\ell(x) = D_{b_s}\psi^\ell(x)$ share these supports for every $s = 0, \dots, m$, the collection of functions $\{\psi_s^\ell : 0 \le \ell \le L, 0 \le s \le m\}$ is an orthonormal system. By simply taking the Fourier transform of Ψ^ℓ and extracting the associated high pass filters $M_\ell(\xi)$, Thm. 3 is satisfied and we have Haar-type composite dilation wavelets.

It is straightforward to observe that the traditional Haar scaling function (2) and wavelet (3) can be obtained in this fashion. For the standard Haar-wavelet, the group $B = \{1\}$ is the trivial group and the lattice is $\mathbb{Z}$. The dilation (matrix) is $a = 2$ and $\varphi(x) = \chi_{[0,1]}(x)$. Therefore,

$$\frac{1}{\sqrt{2}}\varphi\left(\frac{1}{2}x\right) = \frac{1}{\sqrt{2}}\chi_{[0,1]}(x) + \frac{1}{\sqrt{2}}\chi_{[1,2]}(x).$$

The dilated wavelet is obtained from the orthogonal matrix $Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ in the sense that

$$\frac{1}{\sqrt{2}}\psi\left(\frac{1}{2}x\right) = \frac{1}{\sqrt{2}}\chi_{[0,1]}(x) - \frac{1}{\sqrt{2}}\chi_{[1,2]}(x).$$

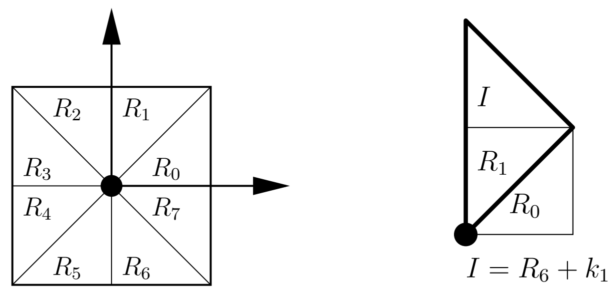
Since defining Haar-type composite dilation wavelets from the composite scaling function and an appropriate orthogonal matrix is effective in general, we define the following orthogonal matrices from which we define CHCDW in Sec. 3. The superscript represents the family of dilation matrices a with a common determinant, namely we have $Q^{|\det(a)|}$. While any such orthogonal matrix will suffice, we list some examples for expanding matrices whose determinants match those of the examples in Sec. 3.

$$Q^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad Q^3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ \frac{\sqrt{6}}{2} & 0 & -\frac{\sqrt{6}}{2} \\ \frac{\sqrt{2}}{2} & -\sqrt{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \qquad Q^4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

$$Q^9 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} & -1 & \frac{1}{2} & \frac{1}{2} & -1 & -1 & 2 \\ \frac{3}{2} & -\frac{3}{2} & 0 & \frac{3}{2} & -\frac{3}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{2\sqrt{2}}{2} & -\frac{2\sqrt{2}}{2} & -\frac{2\sqrt{2}}{2} \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & -\frac{2\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & -\frac{2\sqrt{3}}{2} & 0 & 0 & 0 \\ \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{2} & -\frac{\sqrt{6}}{2} & -\frac{\sqrt{6}}{2} & -\frac{\sqrt{6}}{2} & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{2} & 0 & \frac{\sqrt{6}}{2} & -\frac{\sqrt{6}}{2} & -\frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{2} & 0 \\ -\frac{2\sqrt{30}}{10} & 0 & \frac{2\sqrt{30}}{10} & \frac{2\sqrt{30}}{10} & 0 & -\frac{2\sqrt{30}}{10} & \frac{\sqrt{30}}{10} & -\frac{3\sqrt{30}}{10} & \frac{2\sqrt{30}}{10} \\ -\frac{4\sqrt{5}}{10} & \frac{5\sqrt{5}}{10} & -\frac{\sqrt{5}}{10} & \frac{4\sqrt{5}}{10} & -\frac{5\sqrt{5}}{10} & \frac{\sqrt{5}}{10} & -\frac{8\sqrt{5}}{10} & \frac{4\sqrt{5}}{10} & \frac{4\sqrt{5}}{10} \end{pmatrix}$$

2.5 Example: $B \ltimes \Gamma = p4m$

In [20], a low pass filter matrix was constructed explicitly for the crystallographic group $B \ltimes \Gamma$ by observation of the action of a on a triangular fundamental region, R. In this example $a = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is the quincunx ma-



(a) (b)

Fig. 1 $B \ltimes \Gamma = p4m$: (a) The support sets $R_s = b_s R, s = 0, \ldots 7$ for the scaling functions φ_s; (b) The support sets of the dilated composite scaling function $\varphi_0(a^{-1}x)$ and composite wavelet $\psi_0(a^{-1}x)$ defined by (27) and (30).

trix (which is a counter-clockwise rotation by $\pi/4$ and expansion by $\sqrt{2}$), $B = D_4$ is the group of the full symmetries of the square, and $\Gamma = \mathbb{Z}^2$. In this case $B \ltimes \Gamma = D_4 \ltimes \mathbb{Z}^2$ is the crystallographic group denoted $p4m$ (the interpretation of this group nomenclature is discussed in Sec. 3 or [25]).

Krishtal et al. fix the ordering of B as

$$b_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; b_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; b_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; b_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and $b_i = -b_{i-4}$ for $i = 4, 5, 6, 7$. The scaling function is defined as $\varphi = \frac{1}{\sqrt{\mu(R)}} \chi_R$ with the scaling set, R, defined as the isosceles right triangle with vertices, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$. One may easily verify conditions (i) and (ii) for Thm. 2. See Fig. 1.

To verify condition (iii′), we observe that $aR = b_1 R \cup \left(b_6 R + b_6 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right)$. Thus the index sets are $I = \{1, 6\}$, $P_1 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$, and $P_6 = \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$. Moreover, we observe that this implies

$$D_a \varphi_0(x) = \frac{1}{\sqrt{2}} \varphi_0(a^{-1}x) = \frac{1}{\sqrt{2}} \varphi_1(x) + \frac{1}{\sqrt{2}} T_{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} \varphi_6(x). \tag{27}$$

For s even, b_s is a rotation (or the identity), thus conjugation by a returns b_s. For s odd, b_s is a reflection and conjugation by a returns another reflection in B. Therefore

$$\tilde{s} := \begin{cases} s & \text{for s even,} \\ s + 2 \mod 8 & \text{for s odd.} \end{cases} \tag{28}$$

Furthermore, examining the right orbits of $b_s \in B$ for $0 \leq s \leq m$,

$$s(i) := \begin{cases} s+i \mod 8 & \text{for } s \text{ even,} \\ s-i \mod 8 & \text{for } s \text{ odd.} \end{cases} \tag{29}$$

With (28) and (29), we can construct $M_0(\xi)$ for this example. Let $e(\alpha) = e^{2\pi i\alpha}$ and note that the indices of the $M_0(\xi)$ range from 0 to m. From (27), we can read off the entries of the first row (i.e. the row corresponding to $s = 0$). Since $\det(a) = 2$, the first row has $\frac{1}{2}$ in the second column and $\frac{1}{2}e(-\xi_2)$ in the seventh column, with the remaining entries of the row equal to zero. To further elucidate the notation, we will determine the entries of the second and third row, namely $s = 1, 2$. When $s = 1$, $\tilde{s} = 3$ and therefore $\tilde{s}(1) = 2$ and $\tilde{s}(6) = 5$. Since the entries of $M_0(\xi)$ are zero when $i \notin I$, the only nonzero entries appear in the columns indexed by 2 and 5, i.e. the third and sixth columns. Since P_1 contains only the zero vector, $[M_0(\xi)]_{1,2} = \frac{1}{2}$. Since $b_5 k_{6,0} = b_5\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $[M_0(\xi)]_{1,5} = \frac{1}{2}e(-\xi_2)$. Likewise, with $s = 2$, $\tilde{s} = 2$ and $\tilde{s}(1) = 3, \tilde{s}(6) = 0$. Now $b_{\tilde{s}(1)}k_{1,0} = 0$ and $b_{\tilde{s}(6)}k_{6,0} = b_0\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, therefore we have that $[M_0(\xi)]_{2,3} = \frac{1}{2}$ and $[M_0(\xi)]_{2,0} = \frac{1}{2}e(\xi_1)$. Continuing in this fashion, we reproduce the low pass filter matrix obtained in [20], namely

$$M_0(\xi) = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & e(-\xi_2) & 0 \\ 0 & 0 & 1 & 0 & 0 & e(-\xi_2) & 0 & 0 \\ e(\xi_1) & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & e(\xi_1) \\ 0 & 0 & e(\xi_2) & 0 & 0 & 1 & 0 & 0 \\ 0 & e(\xi_2) & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e(-\xi_1) & 0 & 0 & 1 \\ 1 & 0 & 0 & e(-\xi_1) & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, since $\det(a) = 2$, we can construct the wavelets by either method detailed in Sec. 2.4. Since $\{\beta_0 = (0,0), \beta_1 = (1/2,1/2)\}$ is a full set of coset representatives of $\hat{\mathbb{Z}}^2 a^{-1}/\hat{\mathbb{Z}}^2$, the matrix $M_1(\xi) = M_0(\xi + \beta_1)$ defines a high pass filter matrix:

$$M_1(\xi) = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & -e(-\xi_2) & 0 \\ 0 & 0 & 1 & 0 & 0 & -e(-\xi_2) & 0 & 0 \\ -e(\xi_1) & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -e(\xi_1) \\ 0 & 0 & -e(\xi_2) & 0 & 0 & 1 & 0 & 0 \\ 0 & -e(\xi_2) & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -e(-\xi_1) & 0 & 0 & 1 \\ 1 & 0 & 0 & -e(-\xi_1) & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Taking the first row of the inverse Fourier transform of $\hat{\Psi}^1(\xi) = M_1(\xi)\hat{\Psi}^1(\xi)$ yields the composite dilation wavelet

$$D_a\psi_0(x) = \frac{1}{\sqrt{2}}\psi_0(a^{-1}x) = \frac{1}{\sqrt{2}}\varphi_1(x) - \frac{1}{\sqrt{2}}T_{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)}\varphi_6(x). \tag{30}$$

Obviously, the same wavelet is obtained by selecting the coefficients from the second row of Q^2 and applying them to (27).

3 Crystallographic Haar-type Composite Dilation Wavelets

From the previous section, we have a systematic way to construct crystallographic Haar-type composite dilation wavelets. For a crystallographic group $B \ltimes \Gamma$ (B is finite), we select a fundamental region, R_0, as the possible support set of the scaling function. To ensure that the characteristic function of this set χ_{R_0} will indeed define a scaling function, we must find an expanding matrix a which satisfies the conditions from Thm. 2. We then need to find appropriate coefficients to apply to the scaling equation to define the wavelets as outlined in Sec. 2.4. When this is possible, we have constructed a CHCDW. In Sec. 3.1, we demonstrate the existence of such a system in $\mathbb{R}^n$ by constructing an example with a single wavelet generator. Section 3.2 answers the question almost completely for $\mathbb{R}^2$ by providing a catalog of the groups B which admit such Haar-type CDW.

3.1 A Crystallographic Haar-type Composite Dilation Wavelets for $L^2(\mathbb{R}^n)$

We begin by constructing a singly generated CHCDW in $\mathbb{R}^n$. Let $\Gamma = \mathbb{Z}^n$ and let B be the group generated by the n reflections through the hyperplanes perpendicular to the standard axes in $\mathbb{R}^n$. By construction, $B(\Gamma) = \Gamma$ and thus $B \ltimes \Gamma$ is a crystallographic group in $\mathbb{R}^n$. Now we center the unit hypercube, S, at the origin and take the fundamental region R_0 to be the hypercube with Lebesgue measure 2^{-n} in the all positive orthant, $R_0 = \{x_i \in [0, 1/2] : i = 1, \ldots, n\}$. This is a fundamental region of $B \ltimes \Gamma$ and $S = \cup_{b \in B} bR_0$. Define the scaling function $\varphi(x) = 2^{-n/2}\chi_{R_0}(x)$.

For the expanding matrix, we select the modified permutation matrix

$$a = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 2 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

In this case, a normalizes B, and if B is ordered so that b_n is the reflection through the hyperplane perpendicular to the x_n-axis, then, with the nth standard basis vector denoted $k_n = (0, \ldots, 0, 1)^t$,

$$aR_0 = R_0 \cup (b_n R_0 + k_n)\,.$$

By Thm. 2, φ is a scaling function for an MRA in $L^2(\mathbb{R}^n)$. Since $|\det(a)| = 2$, we can construct the wavelet from the high pass filter $M_1(\xi) = M_0(\xi + \frac{1}{2}k_n^t)$.

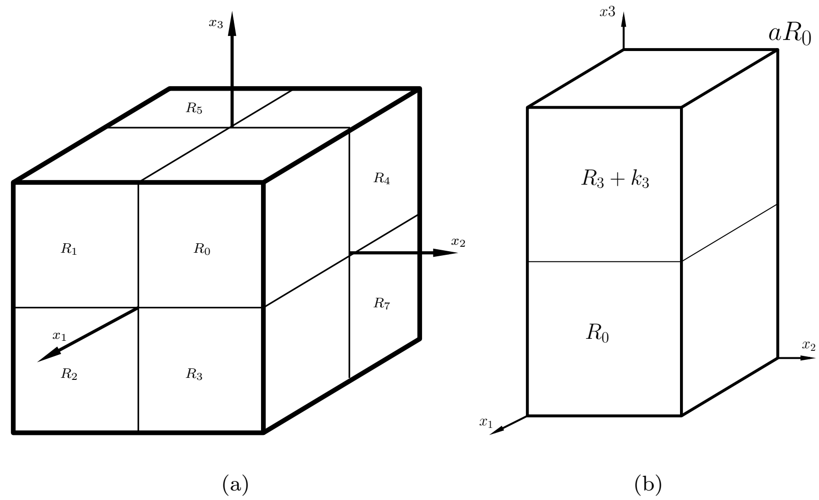


Fig. 2 (a) The support sets $R_s, s = 0, \ldots 7$ for the scaling functions φ_s described in Sec. 3.1 for $\mathbb{R}^3$. (b) The support sets of the dilated composite scaling function $\varphi(a^{-1}x)$ and composite wavelet $\psi(a^{-1}x)$ described in Sec. 3.1 for $\mathbb{R}^3$.

In this case, $\psi(x) = \varphi(ax) - D_{b_n} T_{k_n} \varphi(ax)$. Figure 2 depicts the support sets for this construction in $\mathbb{R}^3$. The matrix filter constructions for this example are more thoroughly treated in [4]. In $\mathbb{R}^2$, this example corresponds to the plane crystallographic group *pmm* in the following section.

3.2 Crystallographic Haar-type Composite Dilation Wavelets for $L^2(\mathbb{R}^2)$

Krishtal et al. [20] showed the existence of simple, nonseparable, Haar-type CDW by providing three important examples. As mentioned in Sec. 2, they did so assuming the crystallographic condition. The preceding discussion has focused on fleshing out the theory of CHCDW and in this section we treat all crystallographic groups which act on $\mathbb{R}^2$. There are 17 crystallographic groups for $\mathbb{R}^2$ which are often referred to as the wallpaper groups, the plane symmetry groups, or the plane crystallographic groups. These groups are well studied and an excellent introduction was written by Schattschneider [25].

In Sec. 3.2.1, we identify the 11 plane crystallographic groups which satisfy the conditions of Thm. 2. In Sec. 3.2.2, we construct support sets for the 11 groups identified in Sec. 3.2.1.

3.2.1 Plane Crystallographic Groups

Table 1 The 17 plane crystallographic groups

Group	Lattice	Factorization	Group	Lattice	Factorization
$p1$	parallelogram	yes	$p4$	square	yes
$p2$	parallelogram	yes	$p4m$	square	yes
pm	rectangular	yes	$p4g$	square	no
pmm	rectangular	yes	$p3$	hexagonal	yes
pg	rectangular	no	$p31m$	hexagonal	yes
pmg	rectangular	no	$p3m1$	hexagonal	yes
pgg	rectangular	no	$p6$	hexagonal	yes
cm	rhombic	yes	$p6m$	hexagonal	yes
cmm	rhombic	yes			

The plane crystallographic groups admit 5 classes of lattices: parallelogram, rectangular, rhombic, square, and hexagonal. These lattices are a key ingredient in the proof that there are only 17 distinct plane crystallographic groups (up to isomorphism). Such a proof can found throughout the literature, for example [1]. There are four basic isometric transformations in a plane crystallographic group: reflection, rotation, translation, and glide reflection. A glide reflection is a translation followed by a reflection across the translation axis. These groups have multiple standard notations assigned to them and we adopt the nomenclature from [25]. The only modifications to the following excerpt from [25] are our notion of a Γ-tiling set, the name of the axis (x_1-axis), and the use of radians for angle measure.

> The interpretation of the full international symbol (read left to right) is as follows: (1) letter p or c denotes primitive or centered Γ-tiling set; (2) integer n denotes the highest order of rotation; (3) symbol denotes a symmetry axis normal to the x_1-axis: m (mirror) indicates a reflection axis, g indicates no reflection, but a glide-reflection, 1 indicates no symmetry axis; (4) symbol denotes a symmetry axis at angle α to x_1-axis, with α dependent on n, the highest order of rotation: $\alpha = \pi$ for $n = 1$ or 2, $\alpha = \frac{\pi}{4}$ for $n = 4$, $\alpha = \frac{\pi}{3}$ for $n = 3$ or 6; the symbols $m, g, 1$ are interpreted as in (3). No symbols in the third and fourth position indicate that the group contains no reflections or glide-reflections.

The 17 plane crystallographic groups are listed in Tab. 1, grouped by lattice type. Table 1 also has a column which specifies if the group can be factored in a semi-direct product $B \ltimes \Gamma$ with B finite. The groups which contain a glide reflection but no other reflection do not have such a factorization and are easily identified by the g which appears in their name. As mentioned briefly before, to fit the composite dilation structure, we form the affine system, $\mathcal{A}_{aB\Gamma}(\Psi) = \{D_a^j D_b T_k \psi^\ell : j \in \mathbb{Z}, b \in B, k \in \Gamma\}$, which clearly requires the action of a group $B \ltimes \Gamma$ in the form of $D_b T_k$ for $(b, k) \in B \ltimes \Gamma$. Therefore, the four crystallographic plane groups without such a factorization, pg, pmg,

Table 2 The factorization of the crystallographic groups into $B \ltimes \Gamma$. The generators for B are rotations and reflections: $\rho(\alpha)$ is a counterclockwise rotation by α; $\sigma(\alpha)$ is a reflection across the line passing through the origin at an angle α with the x_1-axis. The lattice $\Gamma = c\mathbb{Z}^2$ where c is chosen for the appropriate class of lattice: c_1, c_2 are arbitrary, real constants.

Group Factorization

Group	$B = \langle \text{generators} \rangle$	$\Gamma = c\mathbb{Z}^2$	Group	$B = \langle \text{generators} \rangle$	$\Gamma = c\mathbb{Z}^2$
$p1$	$\langle I \rangle$	$c \in GL_n(\mathbb{R})$	$p4$	$\langle \rho(\pi/2) \rangle$	$c = \begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$
$p2$	$\langle \rho(\pi) \rangle$	$c \in GL_n(\mathbb{R})$	$p4m$	$\langle \sigma(0), \sigma(\pi/4) \rangle$	$c = \begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$
pm	$\langle \sigma(0) \rangle$	$c = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$	$p3$	$\langle \rho(2\pi/3) \rangle$	$c = c_1 \begin{pmatrix} \sqrt{3} & 0 \\ 1 & 2 \end{pmatrix}$
pmm	$\langle \sigma(0), \sigma(\pi/2) \rangle$	$c = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$	$p31m$	$\langle \rho(2\pi/3), \sigma(\pi/6) \rangle$	$c = c_1 \begin{pmatrix} \sqrt{3} & 0 \\ 1 & 2 \end{pmatrix}$
cm	$\langle \sigma(0) \rangle$	$c = c_1 \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$	$p3m1$	$\langle \rho(2\pi/3), \sigma(0) \rangle$	$c = c_1 \begin{pmatrix} \sqrt{3} & 0 \\ 1 & 2 \end{pmatrix}$
cmm	$\langle \sigma(0), \sigma(\pi/2) \rangle$	$c = c_1 \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$	$p6$	$\langle \rho(\pi/3) \rangle$	$c = c_1 \begin{pmatrix} \sqrt{3} & 0 \\ 1 & 2 \end{pmatrix}$
			$p6m$	$\langle \sigma(0), \sigma(\pi/6) \rangle$	$c = c_1 \begin{pmatrix} \sqrt{3} & 1 \\ 0 & 2 \end{pmatrix}$

pgg, $p4g$, are not applicable to the composite dilation setting. Using a representation theoretic approach, MacArthur and Taylor constructed wavelets with the groups which do not fit the composite dilation setting [22, 23].

While $p3$ and $p31m$ admit a factorization into $B \ltimes \Gamma$, the nature of the fundamental region for these groups does not seem to allow the introduction of a matrix a which will satisfy the conditions of Thm. 2. Although a proof eludes us that no simple fundamental region can be paired with an expanding matrix a to satisfy the sufficient conditions of Thm. 2, an extensive search for a support set for a single scaling function has failed. For example, with a diagonal expanding matrix a, the expansion of the obvious fundamental regions always requires a translation of only half of one of the sets obtained by applying an element of B to this fundamental region. By permitting two scaling functions, MacArthur [22] identified appropriate support sets which produce the MRA obtained from crystallographic group for which $p3$ or $p31m$ are subgroups with index 2, namely $p6$ and $p6m$. Also, it seems plausible that support sets with fractal structure could be obtained by following the construction of Gröchenig and Madych [10]. Since we seek simple support sets, we omit this analysis.

3.2.2 Plane Crystallographic Haar Wavelets

We now employ the theory from Sec. 2 to construct CHCDW with simple support sets for the remaining 11 plane crystallographic groups. We treat 11 of the 17 since four groups (pg, pmg, pgg, $p4g$) have no factorization into $B \ltimes \Gamma$, and two groups ($p3$, $p3m1$) do not seem to admit simple support sets. The process in the following discussion is to construct the support set of the scaling function, R_0, and then demonstrate that the conditions of Thm. 2 are satisfied when an appropriate dilation matrix a is chosen. By presenting the support sets in Tabs. 3-7, we demonstrate that $S = \cup_{b\in B} bR_0$ is a disjoint union and a Γ-tiling set for $\mathbb{R}^2$ and that aR_0 is a union of translations of the sets $R_s = b_s R_0$. Throughout this section, we fix the ordering of the group $B = \{b_0 = I, b_1, \ldots, b_m\}$ so that the sets $R_s = b_s R_0$ are enumerated in a counterclockwise fashion about the origin to form the Γ-tiling set S.

To form the $L = |\det(a)| - 1$ composite dilation wavelets, we follow the method outlined in Sec. 2.4. Select as coefficients for the characteristic functions the entries of the ℓth row of the matrix $Q^{|\det(a)|}$ to form the dilated wavelet $D_a\psi^\ell(x)$. By applying dilations from B and taking the Fourier transform of the vector-valued wavelets $\Psi^\ell : \mathbb{R}^2 \to \mathbb{R}^{m+1}$, one may readily extract the high pass filter matrices $M_\ell(\xi)$ and verify the conditions of Thm. 3.

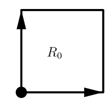
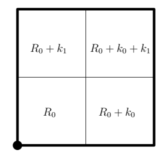
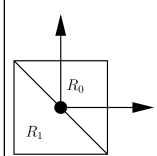
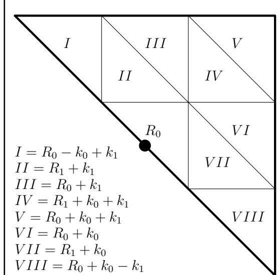
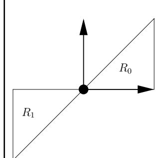
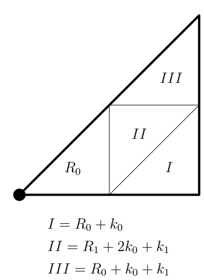
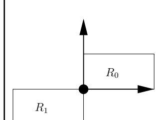
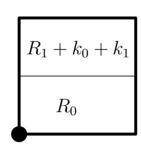
We catalog the CHCDW for these 11 plane crystallographic groups in Tabs. 3-7. The plane crystallographic groups are broken into lattice classes. The square lattice $\mathbb{Z}^2$ is by its nature also a parallelogram and rectangular lattice. These three lattice classes are all presented as a square lattice with the appropriate trivial generalizations (for diagonal dilation matrices) omitted. Tables 3-5 treat the six groups which admit a square lattice. The rhombic lattice applies to two groups, cm and cmm, which are presented in Tab. 6. Finally, $p3m1$, $p6$, and $p6m$ require translations from a hexagonal lattice and are treated in Tab. 7.

There are a few desirable, yet competing, properties to consider: the number of required composite dilation wavelet generating functions, separability of the scaling function, and connectedness of the interior of the Γ-tiling sets. Reducing the number of wavelet generators is certainly desirable and achieved by finding a matrix a with a small determinant. On the other hand, utilizing a dilation matrix of the form $a = \alpha I$ has advantages in implementation by eliminating a possible matrix multiplication. Clearly, this requires $\alpha^2 - 1$ composite dilation wavelet generating functions. Also, the additional structure of the composite dilation setting is most advantageous when the scaling function φ and the wavelets ψ^ℓ are not separable, i.e. they are not products of univariate functions. It is often possible to reduce the number of generators by allowing separable scaling functions and wavelets. Finally, it may be advantageous to have a Γ-tiling set with connected interior. Again, by surrendering the requirement that the interior of $S = \cup_{b\in B} bR_0$ is connected, it sometimes possible to reduce the determinant of a and therefore the necessary number of composite dilation wavelet generating functions.

Tables 3-7 first treat a diagonal matrix $a = nI$ for $n \in \mathbb{Z}$. These are followed, in some cases, by examples where the properties from the previous paragraph are considered. When the support sets are rectangles, this is the case of a separable composite scaling function and composite dilation

Table 3 Construction of support sets for CHCDW. The origin is depicted by a bold dot. The lattice basis elements, namely the columns of the matrix c for $\Gamma = c\mathbb{Z}^n$, are depicted by arrows. The columns with header aR_0 depicts condition (iii) of Thm. 2. The set aR_0 is outlined in bold.

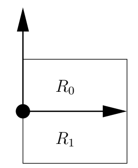
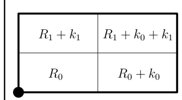
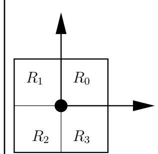
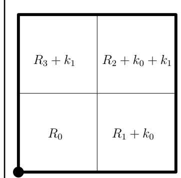
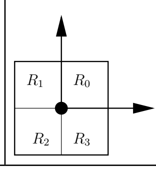
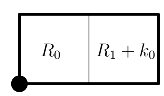
Square Lattice: $\Gamma = \mathbb{Z}^2$

Group	a	Γ-tiling Set	aR_0
$p1$	$2I$		
$p2$	$3I$		
	$2I$		
	$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$		

wavelet. The nonseparable scaling functions and composite dilation wavelets are supported on triangles. Four of these groups ($p1$, $p4m$, $p6$, $p6m$) have already been treated in the literature. Since B is the trivial group for $p1$, this group represents the case of standard wavelets in $\mathbb{R}^2$. The case for a single Haar-type wavelet in $\mathbb{R}^2$ was characterized by Gröchenig and Madych [10] where the support set of the scaling function has a fractal boundary. The geometric complexity of the fractal boundaries does not meet our desire for

Table 4 Construction of support sets for CHCDW. The origin is depicted by a bold dot. The lattice basis elements, namely the columns of the matrix c for $\Gamma = c\mathbb{Z}^n$, are depicted by arrows. The columns with header aR_0 depicts condition (iii) of Thm. 2. The set aR_0 is outlined in bold.

Square Lattice: $\Gamma = \mathbb{Z}^2$

Group	a	Γ-tiling Set	aR_0
pm	$2I$		
pmm	$2I$		
	$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$		

simple support sets and this is omitted from the tables. Krishtal et al. presented three examples of simple Haar-type wavelets [20] for the groups $p4m$, $p6$, and $p6m$. The $p4m$ and $p6m$ examples from [20] appear as the second example for each in Tabs. 5 and 7, respectively. For $p6$, the expanding matrix used in [20] is the product of the counterclockwise rotation by $\pi/3$ and $2I$. Such combinations of a diagonal matrix and a generator for the composite dilation group B (see Tab. 2) are suitable expanding matrices for each of the eleven plane crystallographic groups in Tabs. 3-7.

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Table 5 Construction of support sets for CHCDW. The origin is depicted by a bold dot. The lattice basis elements, namely the columns of the matrix c for $\Gamma = c\mathbb{Z}^n$, are depicted by arrows. The columns with header aR_0 depicts condition (iii) of Thm. 2. The set aR_0 is outlined in bold; when R_0 is not contained in aR_0, R_0 is included to depict the action of a on R_0.

Square Lattice: $\Gamma = \mathbb{Z}^2$

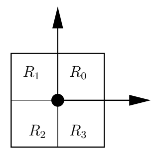
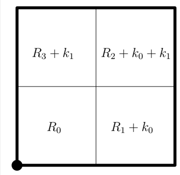
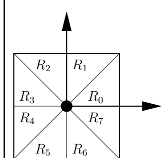
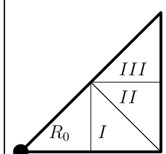
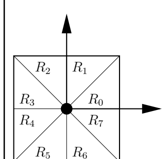
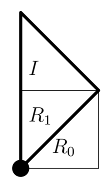
Group	a	Γ-tiling Set	aR_0
$p4$	$2I$		
$p4m$	$2I$		 $I = R_3 + k_0$ $II = R_2 + k_0$ $III = R_5 + k_0 + k_1$
	$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$		 $I = R_6 + k_1$

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Table 6 Construction of support sets for CHCDW. The origin is depicted by a bold dot. The lattice basis elements, namely the columns of the matrix c for $\Gamma = c\mathbb{Z}^n$, are depicted by arrows. The columns with header aR_0 depicts condition (iii) of Thm. 2.

Rhombic Lattice: $\Gamma = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \mathbb{Z}^2$

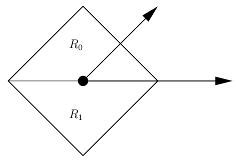
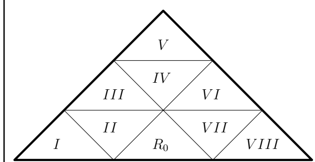
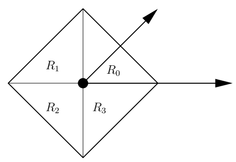
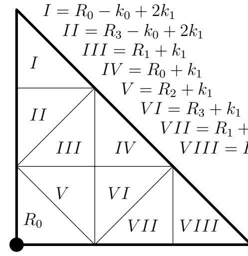
Group	a	Γ-tiling Set	aR_0
cm	$3I$		 $I = R_0 - k_0$, $II = R_1 - k_0 + k_1$, $III = R_0 - k_0 + k_1$, $IV = R_1 - k_0 + 2k_1$, $V = R_0 - k_0 + 2k_1$, $VI = R_0 + k_1$, $VII = R_1 + k_1$, $VIII = R_0 + k_0$
cmm	$3I$		 $I = R_0 - k_0 + 2k_1$, $II = R_3 - k_0 + 2k_1$, $III = R_1 + k_1$, $IV = R_0 + k_1$, $V = R_2 + k_1$, $VI = R_3 + k_1$, $VII = R_1 + k_0$, $VIII = R_0 + k_0$

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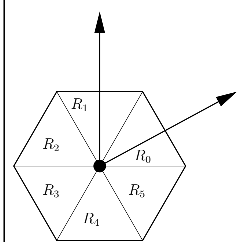
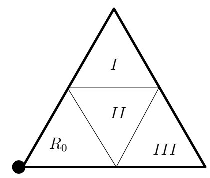
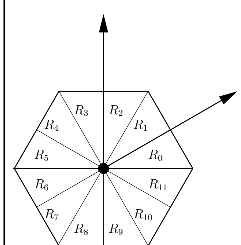
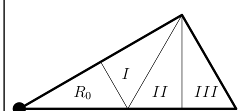
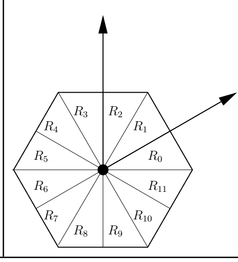
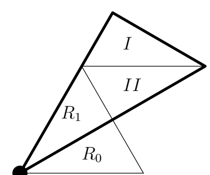
Table 7 Construction of support sets for CHCDW. The origin is depicted by a bold dot. The lattice basis elements, namely the columns of the matrix c for $\Gamma = c\mathbb{Z}^n$, are depicted by arrows. The columns with header aR_0 depicts condition (iii) of Thm. 2. The set aR_0 is outlined in bold; when R_0 is not contained in aR_0, R_0 is included to depict the action of a on R_0.

Hexagonal Lattice: $\Gamma = \frac{3}{4}\begin{pmatrix} \sqrt{3} & 0 \\ 1 & 2 \end{pmatrix}\mathbb{Z}^2$

<table>
<tr><th>Group</th><th>a</th><th>Γ-tiling Set</th><th>aR_0</th></tr>
<tr><td>$p6$
and
$p3m1$</td><td>$2I$</td><td></td><td>
$I = R_2 + k_0$
$II = R_3 + k_0$
$III = R_4 + k_0$</td></tr>
<tr><td rowspan="2">$p6m$</td><td>$2I$</td><td></td><td>
$I = R_7 + k_0$
$II = R_8 + k_0$
$III = R_9 + k_0$</td></tr>
<tr><td>$\frac{\sqrt{3}}{2}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$</td><td></td><td>
$I = R_5 + k_0$
$II = R_6 + k_0$</td></tr>
</table>

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